FRACTIONAL CALCULUS IN AUTOMATICS

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Contens

1. Fractional continuous – time system

$$\sum_{i=0}^{n} A_i \, {}_{t_0}^{C}D_t^{(\nu_i)} y(t) = \sum_{j=0}^{m} B_j \, {}_{t_0}^{C}D_t^{(\mu_j)} u(t)$$

$A_i = const \in \mathbb{R}$, $i = 1,2,\cdots n-1$,

$B_j = const \in \mathbb{R}$, $j = 1,2,\cdots m$,

$m \le n$, $m,n \in \mathbb{Z}_+$,

$A_n = 1$,

$\nu_n > \nu_{n-1} > \cdots > \nu_1 > \nu_0 = 0$,

$\mu_n > \mu_{n-1} > \cdots > \mu_1 > \mu_0 = 0$, $\nu, \mu \in \mathbb{R}_+$,

${}_{t_0}^{C}D_t^{(\nu_i)} y(t)$ - Caputo FD

$y^k(t)\Big|_{t=t_0}, k = 0,1,\cdots,n_{\max}$,

$n_{\max} - 1 \le \nu_n < n_{\max}$

2.1 Fractional transfer function

$$\mathcal{L}\left\{\sum_{i=0}^{p} a_i \; {}_0^C D_t^{(\nu_i)} y(t)\right\} = \mathcal{L}\left\{\sum_{j=0}^{q} a_j \; {}_0^C D_t^{(\mu_j)} u(t)\right\}$$

$$\sum_{i=0}^{n} a_i \, \mathcal{L}\left\{{}_0^C D_t^{(\nu_i)} y(t)\right\} = \sum_{i=0}^{m} b_i \, \mathcal{L}\left\{{}_0^C D_t^{(\nu_i)} u(t)\right\}$$

where $\mathcal{L}\{y(t)\} = Y(s)$, $\mathcal{L}\{u(t)\} = U(s)$

$$G(s) = \frac{Y(s)}{U(s)} =$$

$$\frac{b_q s^{\mu_q} + b_{q-1} s^{\mu_{q-1}} + \cdots + b_1 s^{\mu_1} + b_0}{s^{\nu_p} + a_{p-1} s^{\nu_{p-1}} + \cdots + a_1 s^{\nu_1} + a_0}$$

$0 < \mu_1 < \mu_2 < \cdots < \mu_q$,

$0 < \nu_1 < \nu_2 < \cdots < \nu_p$,

$\mu_q < \nu_p$

$a_i, b_i \in \mathrm{R}$

Characterisitc equation

$$s^{\nu_p} + a_{p-1} s^{\nu_{p-1}} + \cdots + a_1 s^{\nu_1} + a_0 = 0$$

Example 2.1

$$G(s)=\frac{Y(s)}{U(s)}=\frac{3s^{0.5}+4}{s^{1.5}+2s+1}$$

$$G(s)=\frac{Y(s)}{U(s)}=\frac{2}{s+2s^{0.5}+1}$$

$$G(s)=\frac{Y(s)}{U(s)}=\frac{1}{s+1}=\frac{1}{s^{0.5}+j}\cdot\frac{1}{s^{0.5}-j}$$

2.1.1 Impulse response of a fractional system described by a transfer function

One assumes that $\nu_1, \nu_2, \cdots, \nu_p$, $\mu_1, \mu_2, \cdots, \mu_q$ are national numbers

$$\nu_i = \frac{e_i}{d_i} \text{ for } i = 1,2,\cdots,p,$$

$$\mu_i = \frac{g_i}{f_i} \text{ for } i = 1,2,\cdots,q,$$

n - least common denominator

$$\nu_i = \frac{n_i}{n} \text{ for } i = 1,2,\cdots,p,$$

$$\mu_i = \frac{m_i}{n} \text{ for } i = 1,2,\cdots,q.$$

For $n_i, m_i, n \in \mathrm{Z}_+ \cup \{0\}$

$$G(s)=$$

$$\frac{b_q\left(s^{\frac{1}{n}}\right)^{m_q}+b_{q-1}\left(s^{\frac{1}{n}}\right)^{m_{q-1}}+\cdots+b_1\left(s^{\frac{1}{n}}\right)^{n_1}+b_0}{\left(s^{\frac{1}{n}}\right)^{n_p}+a_{p-1}\left(s^{\frac{1}{n}}\right)^{n_{p-1}}+\cdots+a_1\left(s^{\frac{1}{n}}\right)^{n_1}+a_0}$$

Introducing a new variable

$$w=s^{\frac{1}{n}}$$

one gets

$$F(w)=\frac{b_q w^{m_q}+b_{q-1}w^{m_{q-1}}+\cdots+b_1 w^{m_1}+b_0}{w^{n_p}+a_{p-1}w^{n_{p-1}}+\cdots+a_1 w^{n_1}+a_0}$$

$$\sum_{i=1}^{n_{p,R}}\frac{R_i}{w+w_{i,R}}+$$

$$\sum_{i=1}^{n_{p,C}}\left(\frac{C_i}{w+w_{i,C}}+\frac{C_i^*}{w+w_{i,C}^*}\right)$$

$n_{p,R}$ - number of real poles,

$n_{p,R}$ $w_{i,R}$ - real pole,

$n_{p,C}$ - number of complex poles,

$w_{i,C}$ - complex pole,

R_i - real coefficients,

C_i - complex coefficient.

Example 2.2

Find an impulse response of a system

$$G(s) = \frac{0.75}{s\left(s^{\frac{3}{2}} + 3s^{\frac{2}{2}} + 2.75s^{\frac{1}{2}} + 0.75\right)}$$

$$G(s) =$$

$$-\frac{\frac{11}{3}}{s^{0.5}} + \frac{1}{s} + \frac{6}{s^{0.5} + \frac{1}{2}} - \frac{3}{s^{0.5} + 1} + \frac{\frac{2}{3}}{s^{0.5} + \frac{3}{2}}$$

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \sum_{i=1}^{5} f_i(t)$$

$$f_1(t) =$$

$$\mathcal{L}^{-1}\left\{\frac{-\frac{11}{3}}{s^{0.5}}\right\} = \frac{-\frac{11}{3}}{\Gamma(0.5)} t^{-0.5} = -\frac{11}{3\Gamma(0.5) t^{0.5}}$$

$$f_2(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$f_3(t) = \mathcal{L}^{-1}\left\{\frac{6}{s^{0.5} + \frac{1}{2}}\right\} = \frac{6}{t^{0.5}} E_{1,0.5}\left(\frac{t}{4}\right) - 3e^{\frac{t}{4}}$$

$$f_4(t) = \mathcal{L}^{-1}\left\{\frac{-3}{s^{0.5} + 1}\right\} = -\frac{3}{t^{0.5}} E_{1,0.5}(t) + 3e^{t}$$

$$f_5(t) =$$

$$\mathcal{L}^{-1}\left\{\frac{\frac{2}{3}}{s^{0.5}+\frac{3}{2}}\right\} = \mathcal{L}\,\frac{2}{3t^{0.5}}\,E_{1,0.5}\left(\frac{9t}{4}\right) - e^{\frac{9t}{4}}$$

$$E_{1,\frac{1}{2}}\left(\frac{t}{4}\right) = \sum_{i=0}^{\infty}\frac{\left(\frac{t}{4}\right)^{i}}{\Gamma\left(\frac{1}{2}+i\right)}$$

$$E_{1,\frac{1}{2}}(t) = \sum_{i=0}^{\infty}\frac{(t)^{i}}{\Gamma\left(\frac{1}{2}+i\right)}$$

$$E_{1,0.5}\left(\frac{9t}{4}\right) = \sum_{i=0}^{\infty}\frac{\left(\frac{9t}{4}\right)^{i}}{\Gamma\left(\frac{1}{2}+i\right)}$$

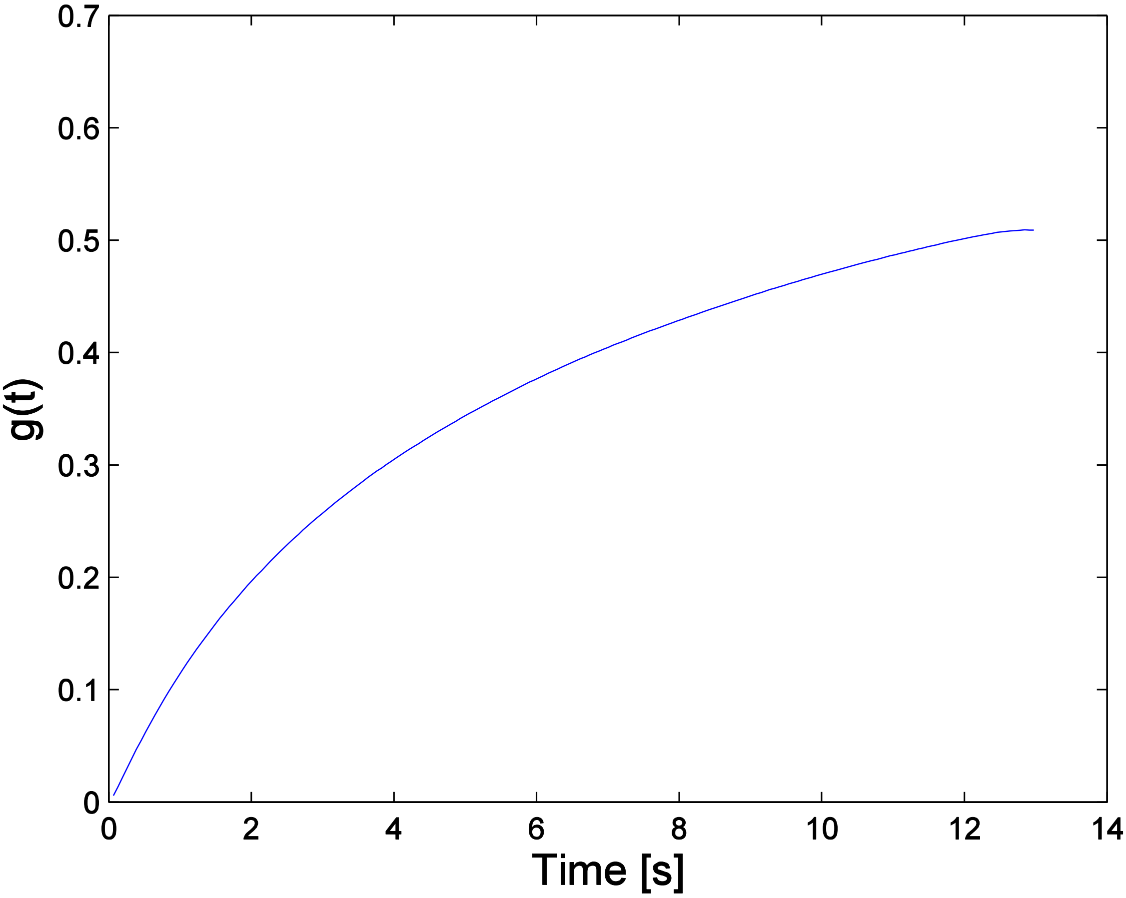

Fig.2.1 Impulse response of a fractional system

2.2 State-space description of a fractional system

$$ {}_0^C D_t^{(\nu)} \mathbf{x}(t) = \mathbf{f}[\mathbf{x}(t), u(t)] $$

$$ \boldsymbol{\nu} = \begin{bmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_n \end{bmatrix} \text{ where } 0 < \nu_1 \le \nu_2 \le \cdots \le \nu_n \le 1 $$

$n \times 1$- vector of fractional orders,

$0 < \nu_1 < \nu_2 < \cdots < \nu_n \le 1$

non-commensurate orders

$0 < \nu_1 = \nu_2 = \cdots = \nu_n \le 1$

commensurate orders

$$\mathbf{x}(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

$n\times 1$ – state vector

$$ {}_0^C D_t^{(\nu)}\mathbf{x}(t) = {}_0^C D_t^{(\nu)}\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} {}_0^C D_t^{(\nu_1)} x_1(t) \\ {}_0^C D_t^{(\nu_2)} x_2(t) \\ \vdots \\ {}_0^C D_t^{(\nu_n)} x_n(t) \end{bmatrix}$$

$$\mathbf{f}(t)=\begin{bmatrix} f_1[\mathbf{x}(t),u(t)] \\ f_2[\mathbf{x}(t),u(t)] \\ \vdots \\ f_n[\mathbf{x}(t),u(t)] \end{bmatrix} = \begin{bmatrix} f_1[x_1(t),\cdots x_n(t),u(t)] \\ f_2[x_1(t),\cdots x_n(t),u(t)] \\ \vdots \\ f_n[x_1(t),\cdots x_n(t),u(t)] \end{bmatrix}$$

SISO linear, time – invariant continuous – time commensurate FO system

$$
{}_0^C D_t^{(v)} \mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t)
$$
$$
y(t) = \mathbf{c}\mathbf{x}(t) + du(t)
$$

$$
\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n-1} & a_{1n} \\ \vdots & & \vdots & \vdots \\ a_{n-11} & \cdots & a_{n-1n-1} & a_{n-1n} \\ a_{n1} & \cdots & a_{nn-1} & a_{nn} \end{bmatrix},
$$

$$
\mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}, \mathbf{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix}^{\mathrm{T}}, d = [d_1]
$$

$$
0 < v_1 = v_2 = \cdots = v_n = v < 1
$$

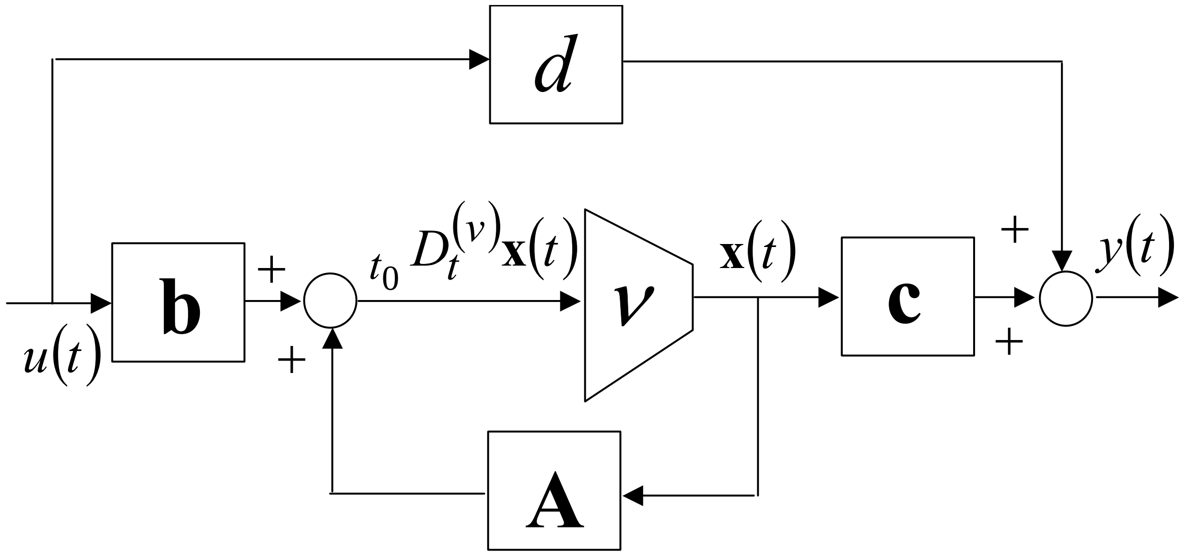

Fig.2.1 Block diagram of the FO linear system with the FO integrator.

2.2.1 Response of a fractional system described by a state space equations

For $\mathbf{x}(0)$ and $u(\mathrm{t})$

$$\mathbf{x}(t) = \Phi_0(t)\mathbf{x}(0) + \int_0^t \Phi(t-\tau)\mathbf{B}u(\tau)d\tau$$

where

$$\Phi_0(t) = E_{\nu,1}\left(\mathbf{A}t^{\nu}\right) = \sum_{i=0}^{\infty} \frac{\mathbf{A}^i t^{i\nu}}{\Gamma(i\nu+1)}$$

$$\Phi(t) = \sum_{i=0}^{\infty} \frac{\mathbf{A}^i t^{(i+1)\nu-1}}{\Gamma[(i+1)\nu]}$$

Example 2.3

Evaluate FO system described by the state – space equations a homogenous response for

$$\mathbf{A} = \begin{bmatrix} 0.1578 & 0.0289 \\ -1.1156 & 0.0422 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \mathbf{c} = \begin{bmatrix} 1 & 1 \end{bmatrix},$$

$$d = [0], \ \mathbf{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Eigenvalues of a matrix **A**

$$s_1 = 0.10 + j0.17$$

$$s_2 = 0.10 - j0.17$$

$$\mathrm{Re}\{s_2\} = 0.10 > 0$$

Asymptotically stable system.

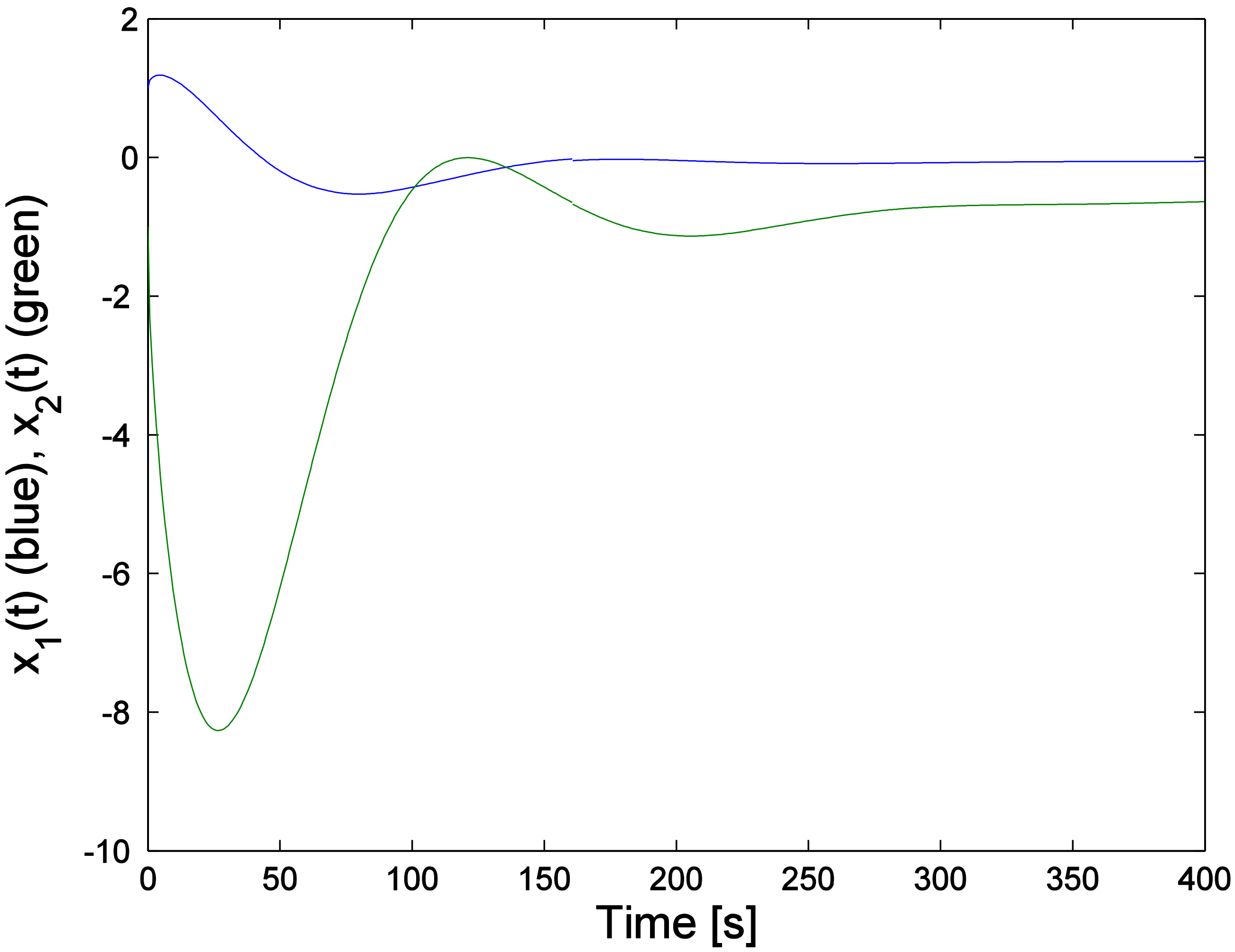

Fig.2.2 FO system states vs. time.

2.3 Frequency characteristics of fractional elements

$$G(j\omega) = G(s)\big|_{s=j\omega}$$

$$s = j\omega = \omega e^{j\frac{\pi}{2}} = \omega\left[\cos\left(\frac{\pi}{2}\right) + j\sin\left(\frac{\pi}{2}\right)\right]$$

$$s^{\nu} = (j\omega)^{\nu} = \omega e^{j\nu\frac{\pi}{2}} =$$

$$\omega^{\nu}\left[\cos\left(\nu\frac{\pi}{2}\right) + j\sin\left(\nu\frac{\pi}{2}\right)\right]$$

3.1 Frequency characteristics of the fractional integrator (FI)

FI transfer function

$$G_I(s)=\frac{Y(s)}{U(s)}=\frac{1}{(T_u s)^{\nu}}=\frac{1}{\left(\frac{s}{\omega_u}\right)^{\nu}}$$

Nyquist charasteristics of the FI

$$Q_I^{(\nu)}=-tg\left(\frac{\pi\nu}{2}\right)P_I^{(\nu)}$$

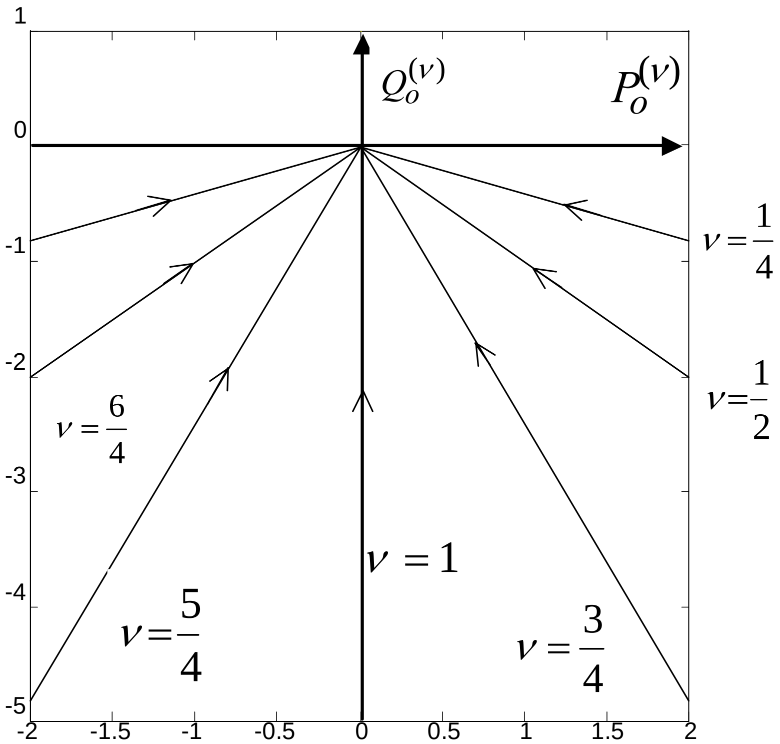

Fig.2.3 Nyquist charastristics of the FI for different orders and constant T_u

The amplitude characteristic of the FI

$$\left|G_I^{(\nu)}(j\omega)\right| =$$

$$\sqrt{\left[P_I^{(\nu)}(j\omega)\right]^2 + \left[Q_I^{(\nu)}(j\omega)\right]^2} = \frac{1}{(T_u\omega)^\nu}$$

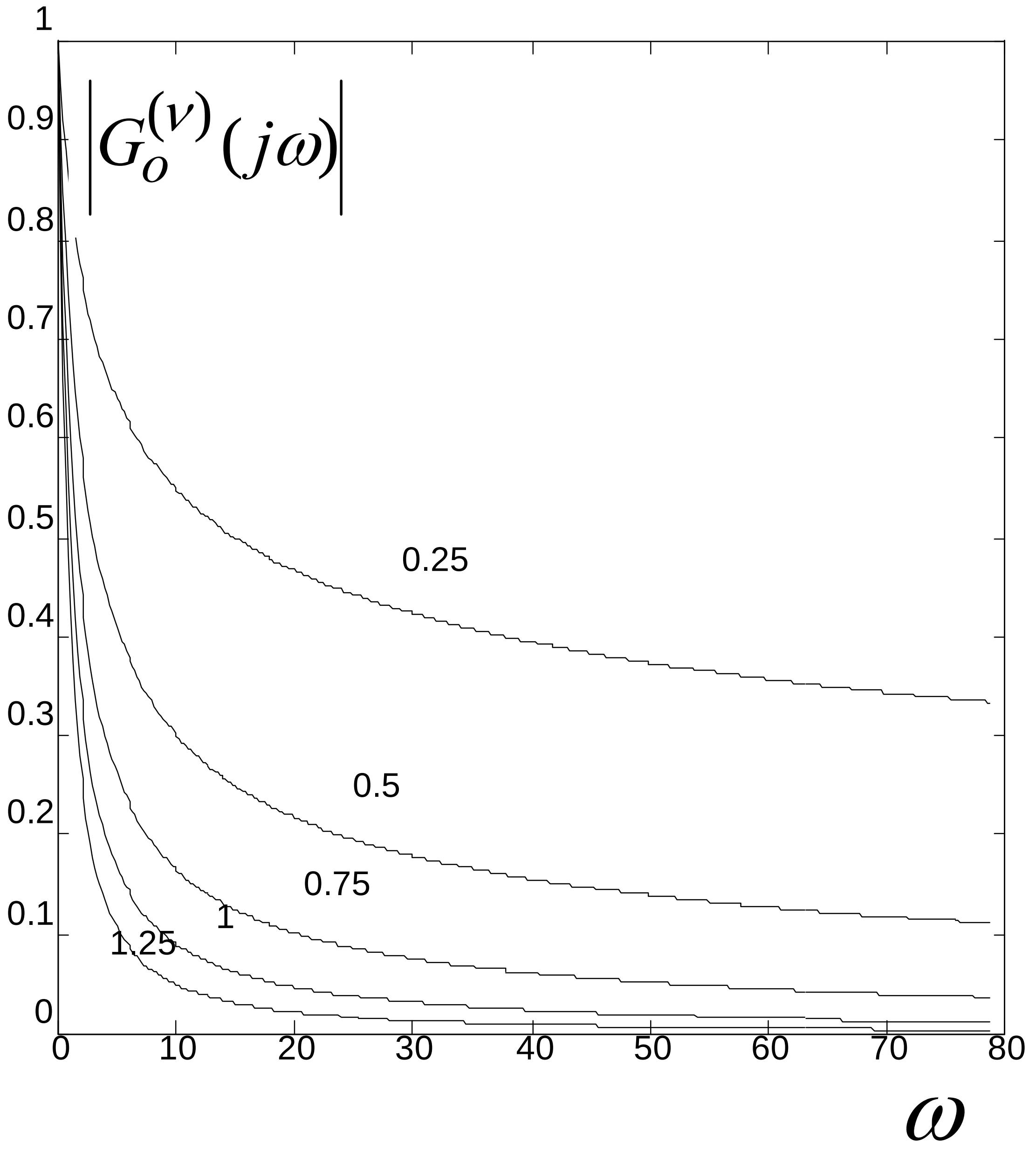

Fig.2.4 Amplitude characteristic of the FI for different orders and constant T_u

The phase angle characteristic of the FI

$$\varphi(\omega) = \operatorname{arctg}\left(\frac{Q_I^{(\nu)}(j\omega)}{P_I^{(\nu)}(j\omega)}\right) =$$

$$-\operatorname{arctg}\left(\operatorname{tg}\left(\frac{\pi\nu}{2}\right)\right) = -\frac{\pi\nu}{2}$$

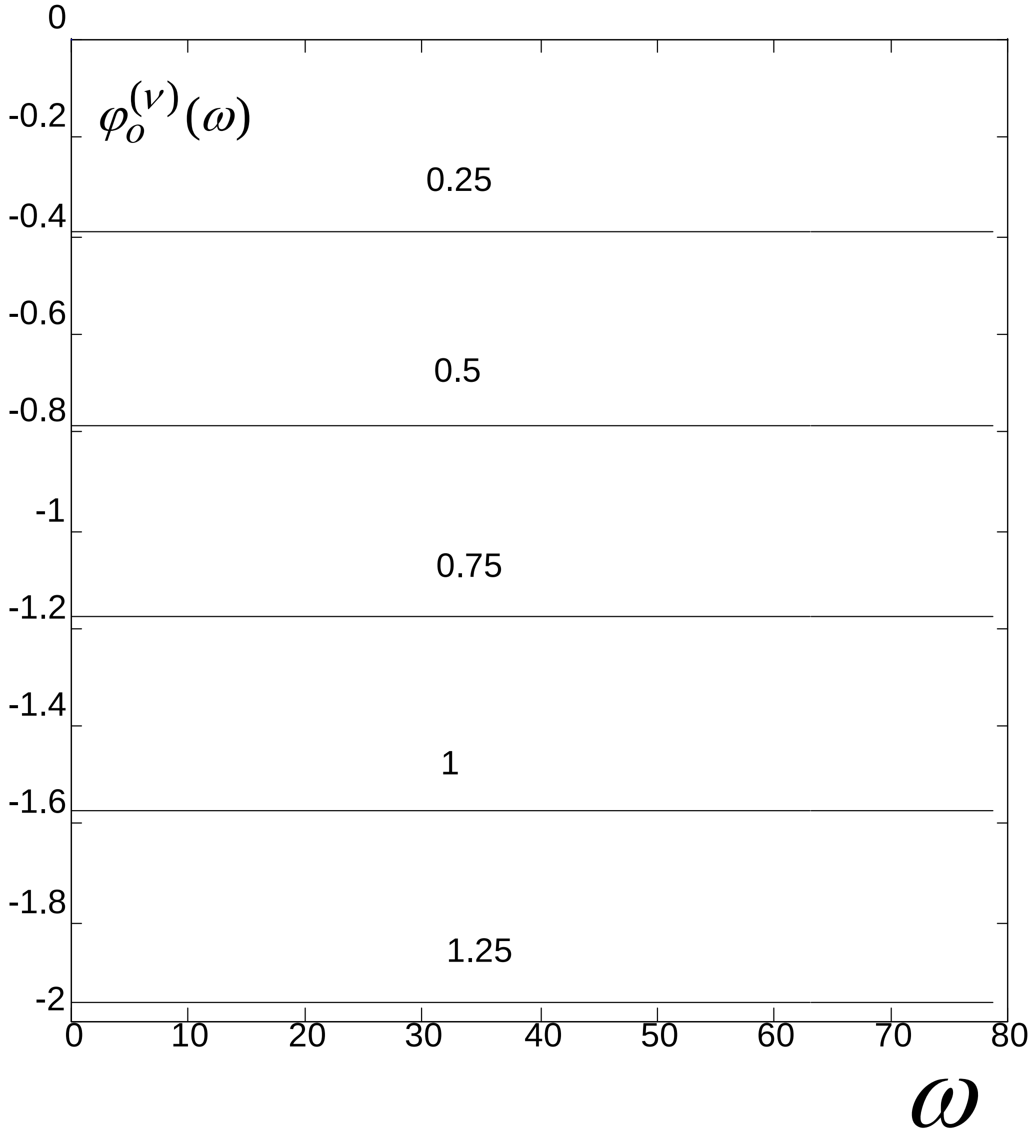

Fig.2.5 Phase angle characteristic of the FI for different orders and constant T_u

3.2 Frequency characteristics of the FO inertial element (FOIE)

$$0 < \nu \leq 2$$

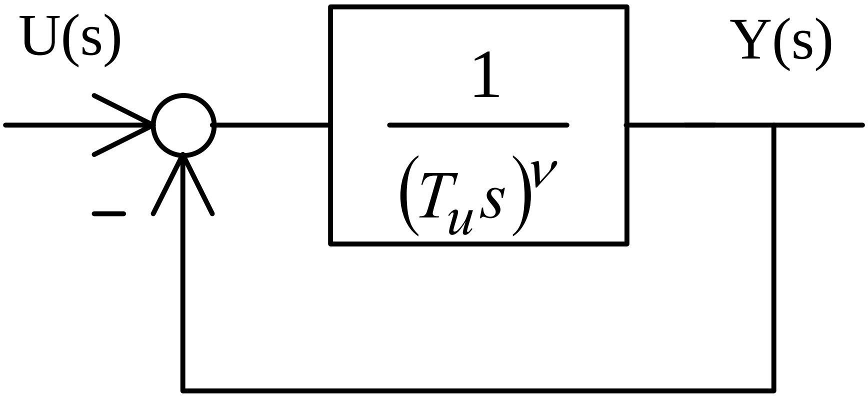

Fig.2.6 Block diagram of the FO inertial element with the time constant T_u

$$G_c^{(\nu)}(s) = \frac{1}{(T_u s)^\nu + 1} = \frac{1}{\left(\frac{s}{\omega_u}\right)^\nu + 1}$$

FOIE Nyquist characteristic equation

$$\left[P_c^{(\nu)}-\frac{1}{2}\right]^2+\left[Q_c^{(\nu)}-\frac{1}{2}\operatorname{ctg}\left(\nu\frac{\pi}{2}\right)\right]^2=\left(\frac{1}{2\sin\left(\nu\frac{\pi}{2}\right)}\right)^2$$

Nyquist plots of the FOIE

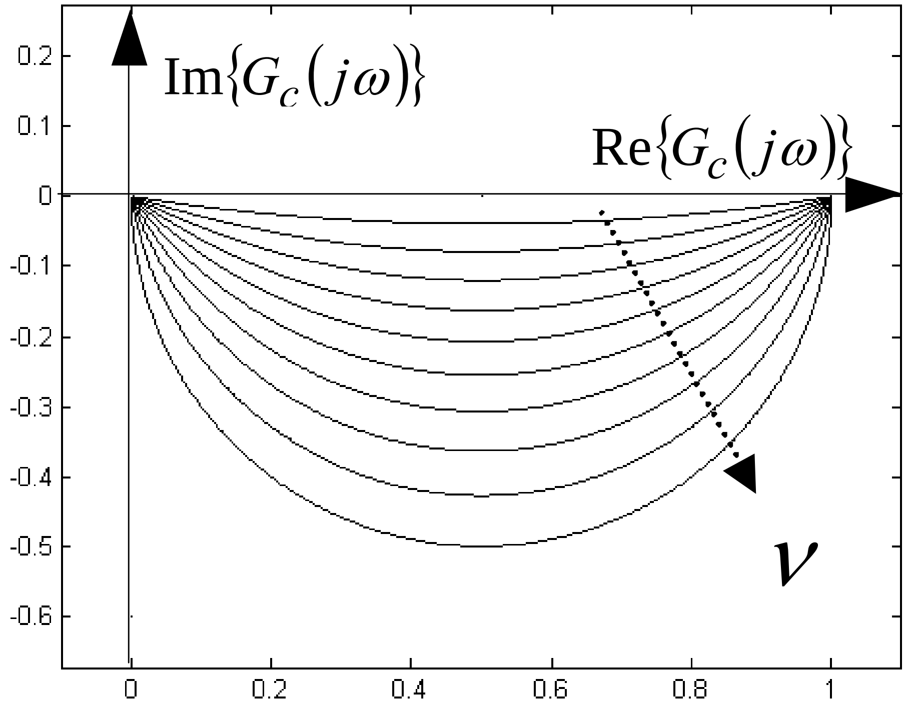

Fig.2.7 Nyquist plots of FOIE for $\nu \in [0.1, 0.2, \cdots 0.9, 1.0]$

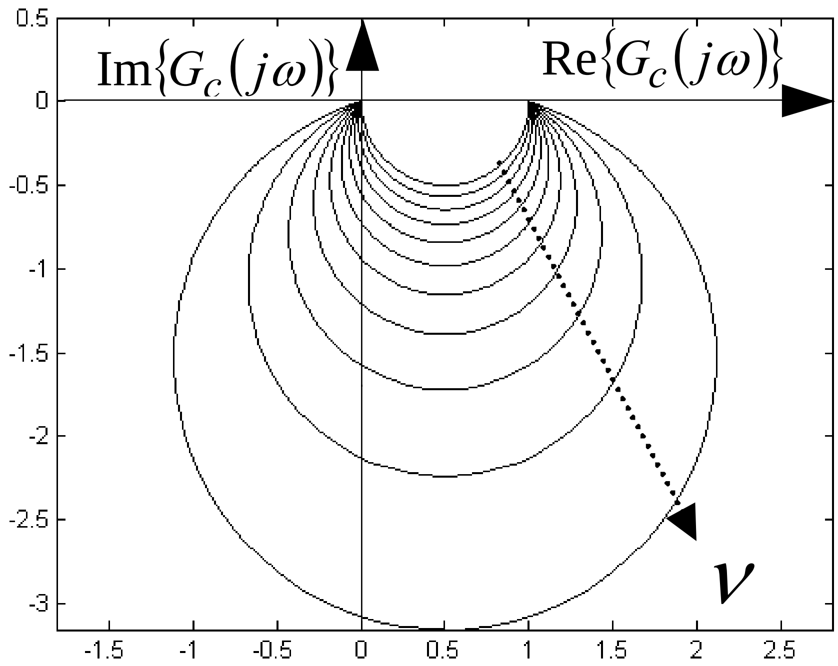

Fig.2.8 Nyquist plots of FOIE for $\nu \in [1.00, 1.08, \cdots 1.72, 1.80]$

The amplitude characteristic of the FOIE

$$\left|G_c(j\omega)\right| = \frac{1}{\left(\frac{\omega}{\omega_u}\right)^{2\nu} + 2\cos\left(\nu\frac{\pi}{2}\right)\left(\frac{\omega}{\omega_u}\right)^{\nu} + 1}$$

The phase angle characteristic of the FOIE

$$\varphi_c(\omega) = -\mathrm{arctg}\left[\frac{\left(\frac{\omega}{\omega_u}\right)^{\nu}\sin\left(\nu\frac{\pi}{2}\right)}{1+\left(\frac{\omega}{\omega_u}\right)^{\nu}\cos\left(\nu\frac{\pi}{2}\right)}\right]$$

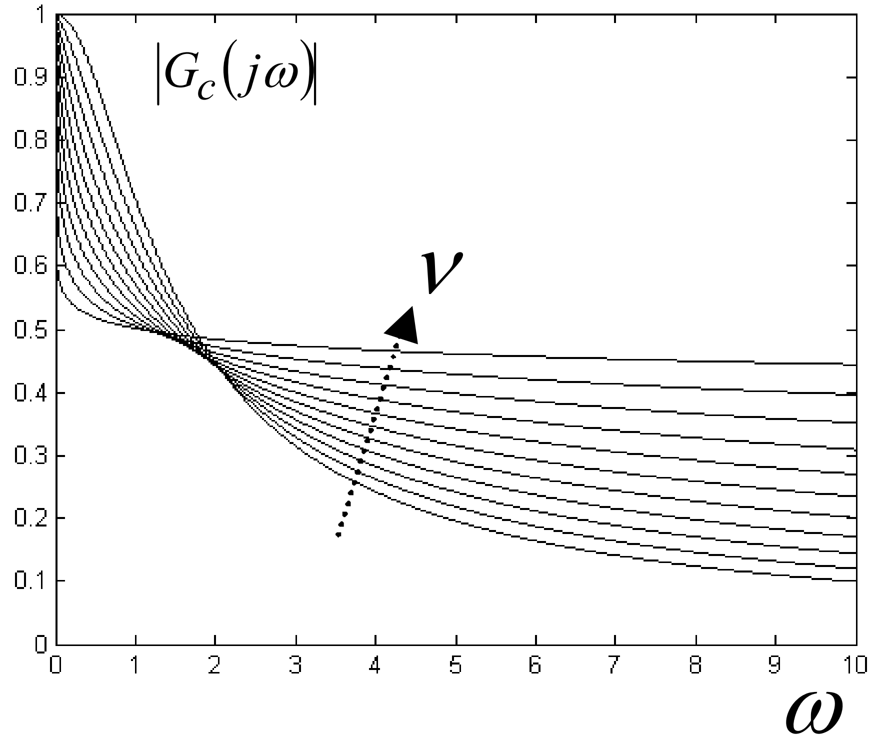

Fig.2.9 Amplitude characteristic of the FOIE for orders $\nu \in [0.1, 0.2, \cdots 0.9, 1.0]$ and constant T_u

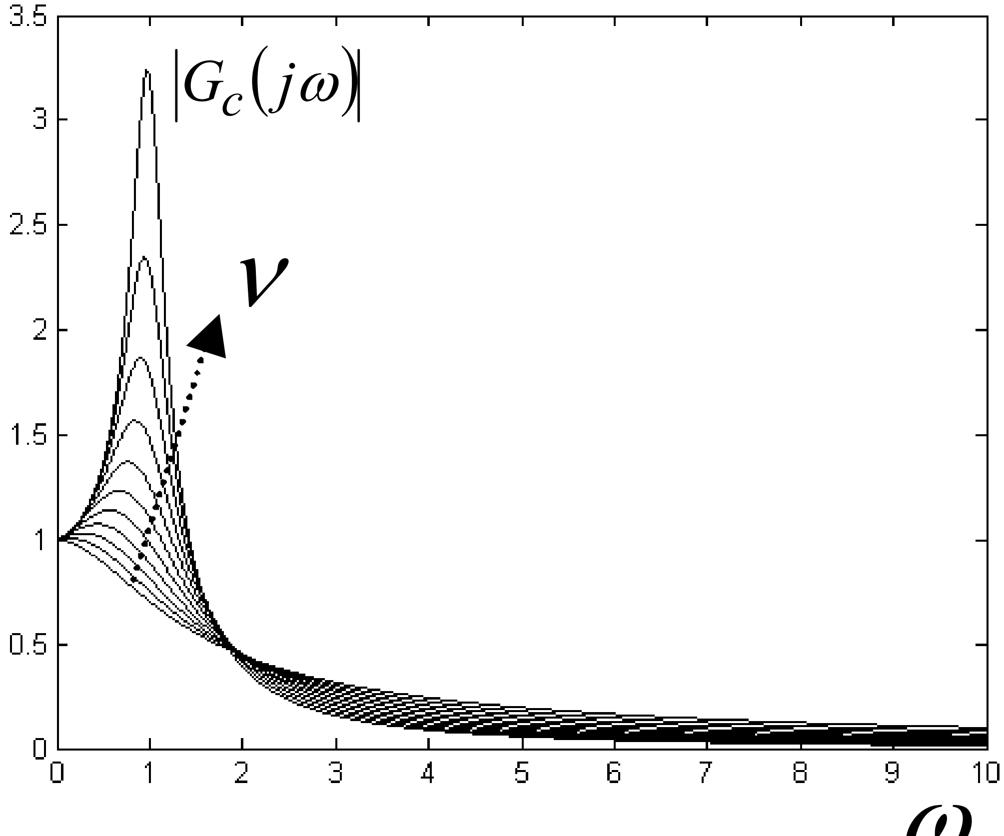

Fig.2.10 Amplitude characteristic of the FOIE for orders $\nu \in [1.1, 1.2, \cdots 1.8, 1.9]$ and constant T_u.

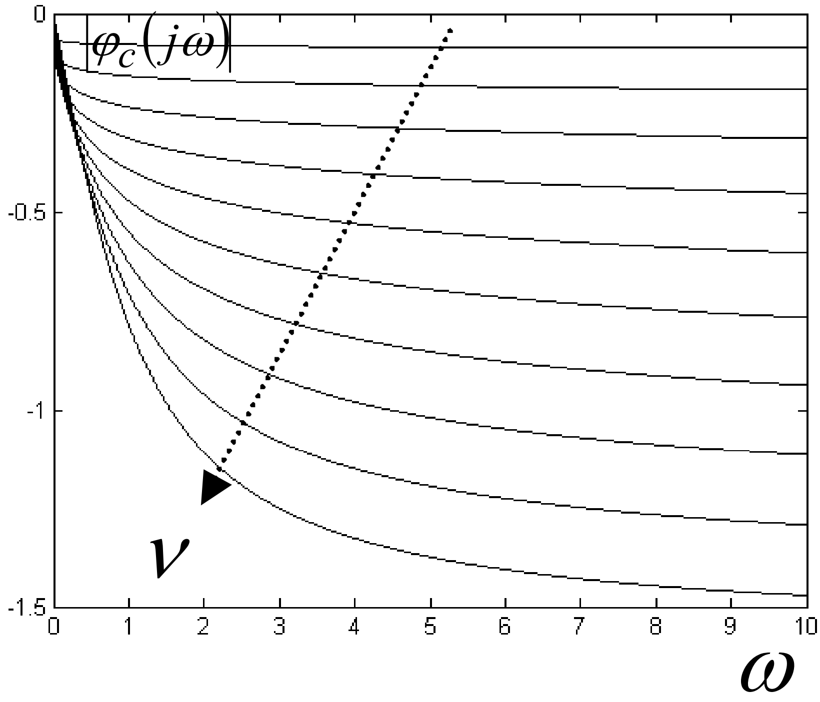

Fig.2.11 Phase angle characteristic of the FI for $\nu \in [0.1, 0.2, \cdots 0.9, 1.0]$ and constant T_u .

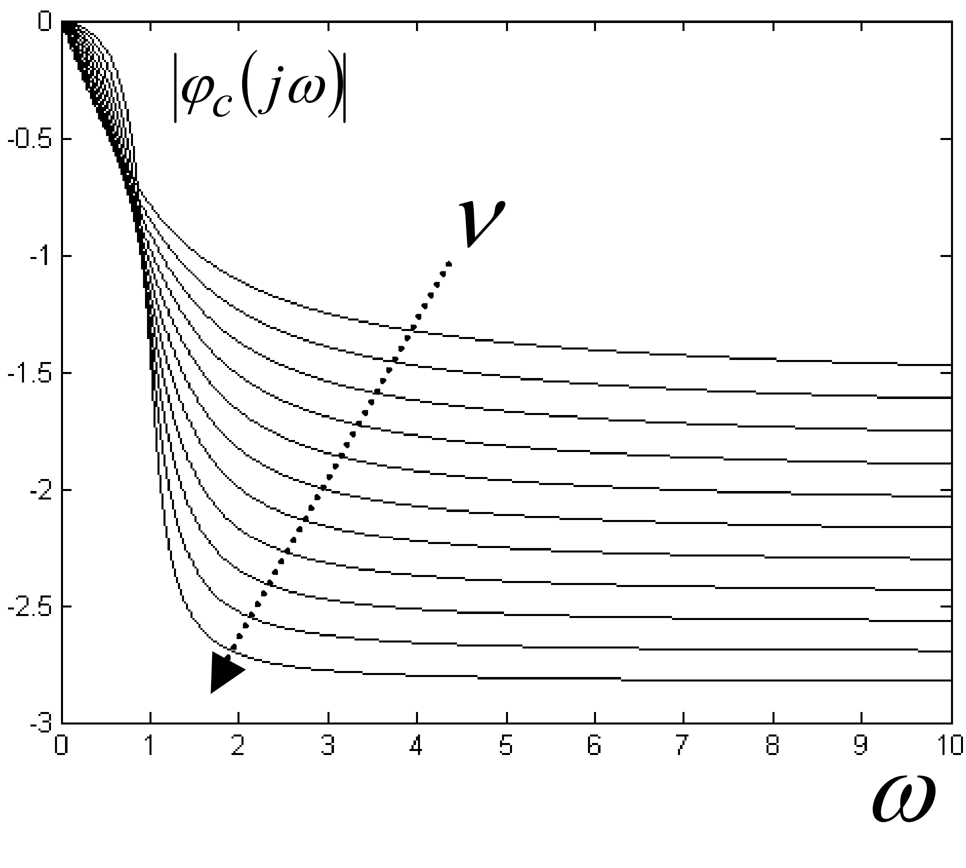

Fig.2.12 Phase angle characteristic of the FI for $\nu \in [1.1, 1.2, \cdots 1.8, 1.9]$ and constant T_u .

2.4Realisations of a fractional system

Consider a commensurate system described by a transfer function

$$G(s)=\frac{Y(s)}{U(s)}=$$

$$\frac{b_q\left(s^{\nu}\right)^{q-p}+b_{q-1}\left(s^{\nu}\right)^{q-p-1}+\cdots+b_1\left(s^{\nu}\right)^{1-p}+b_0\left(s^{\nu}\right)^{-p}}{1+a_{p-1}\left(s^{\nu}\right)^{-1}+\cdots+a_1\left(s^{\nu}\right)^{1-p}+a_0\left(s^{\nu}\right)^{-p}}=$$

$$=\frac{b_q\left(\frac{1}{s^{\nu}}\right)^{p-q}+b_{q-1}\left(\frac{1}{s^{\nu}}\right)^{p-q-1}+\cdots+b_1\left(\frac{1}{s^{\nu}}\right)^{p-1}+b_0\left(\frac{1}{s^{\nu}}\right)^{p}}{1+a_{p-1}\left(\frac{1}{s^{\nu}}\right)^{1}+\cdots+a_1\left(\frac{1}{s^{\nu}}\right)^{p-1}+a_0\left(\frac{1}{s^{\nu}}\right)^{p}}$$

$$s^{\nu}=s^{\frac{1}{d}}$$

d - least common denominator of all orders

$$\mathbf{x}(t)=\begin{bmatrix} x_1(t) \\ \vdots \\ x_{p-1}(t) \\ x_p(t) \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -a_{p-1} & -a_{p-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \mathbf{c} = \begin{bmatrix} 0 & \cdots & 0 & b_q & b_{q-1} & \cdots & b_0 \end{bmatrix},$$

$$d = [0].$$

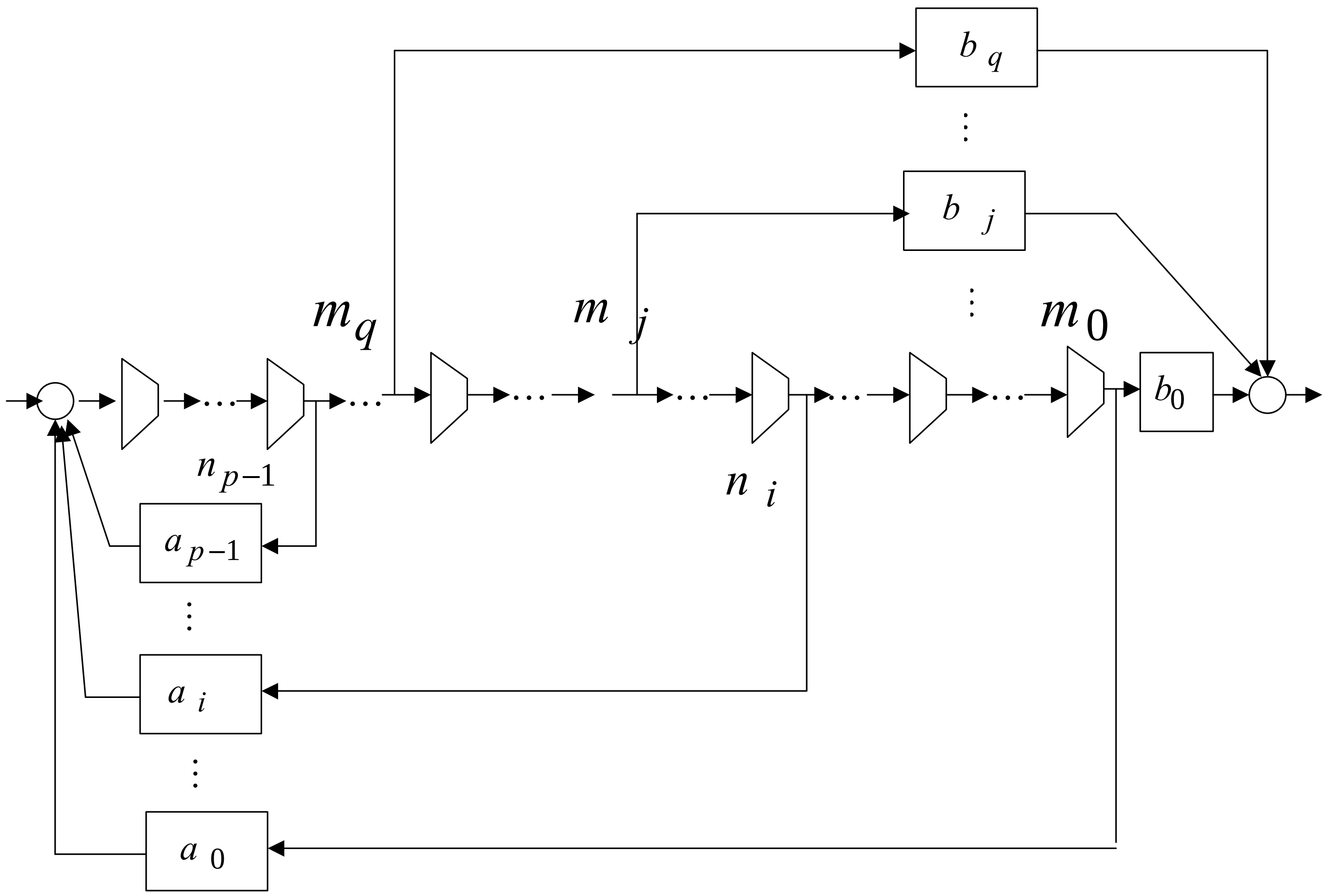

Fig. 2.13 State – space realisation with FOIs of order ν

$$G(s) = \frac{Y(s)}{U(s)} = \mathbf{c}\left[s^{\nu}\mathbf{1} - \mathbf{A}\right]^{-1}\mathbf{b} + d$$

2.5Stability of a fractional system

Characteristic polynomial

$$D\left(s^{\nu}\right)=\det\left[s^{\nu}\mathbf{I}-\mathbf{A}\right] =$$

$$\left(s^{\nu}\right)^{n_p}+a_{p-1}\left(s^{\nu}\right)^{n_{p-1}}+\cdots+a_1\left(s^{\nu}\right)^{n_1}+a_0$$

$$0<\nu=\frac{1}{n}<1$$

$$\sigma=s^{\nu}$$

Characteristic equation

$$D(\sigma)=\det[\sigma\mathbf{I}-\mathbf{A}]=$$

$$\sigma^{n_p}+a_{p-1}\sigma^{n_{p-1}}+\cdots+a_1\sigma^{n_1}+a_0=$$

$$(\sigma-\sigma_1)(\sigma-\sigma_2)\cdots\left(\sigma-\sigma_{n_p}\right)$$

Theorem

Linear, time - inwariant FO dynamic system is asymptotically stable if and only if

$$\left|\arg\{\sigma_i\}\right| > \nu\frac{\pi}{2} = \alpha \text{ for } i = 1,2,\cdots,n_p$$

σ_i for $i = 1,2,\cdots,n_p$ may be real or complex , distinct or multiple.

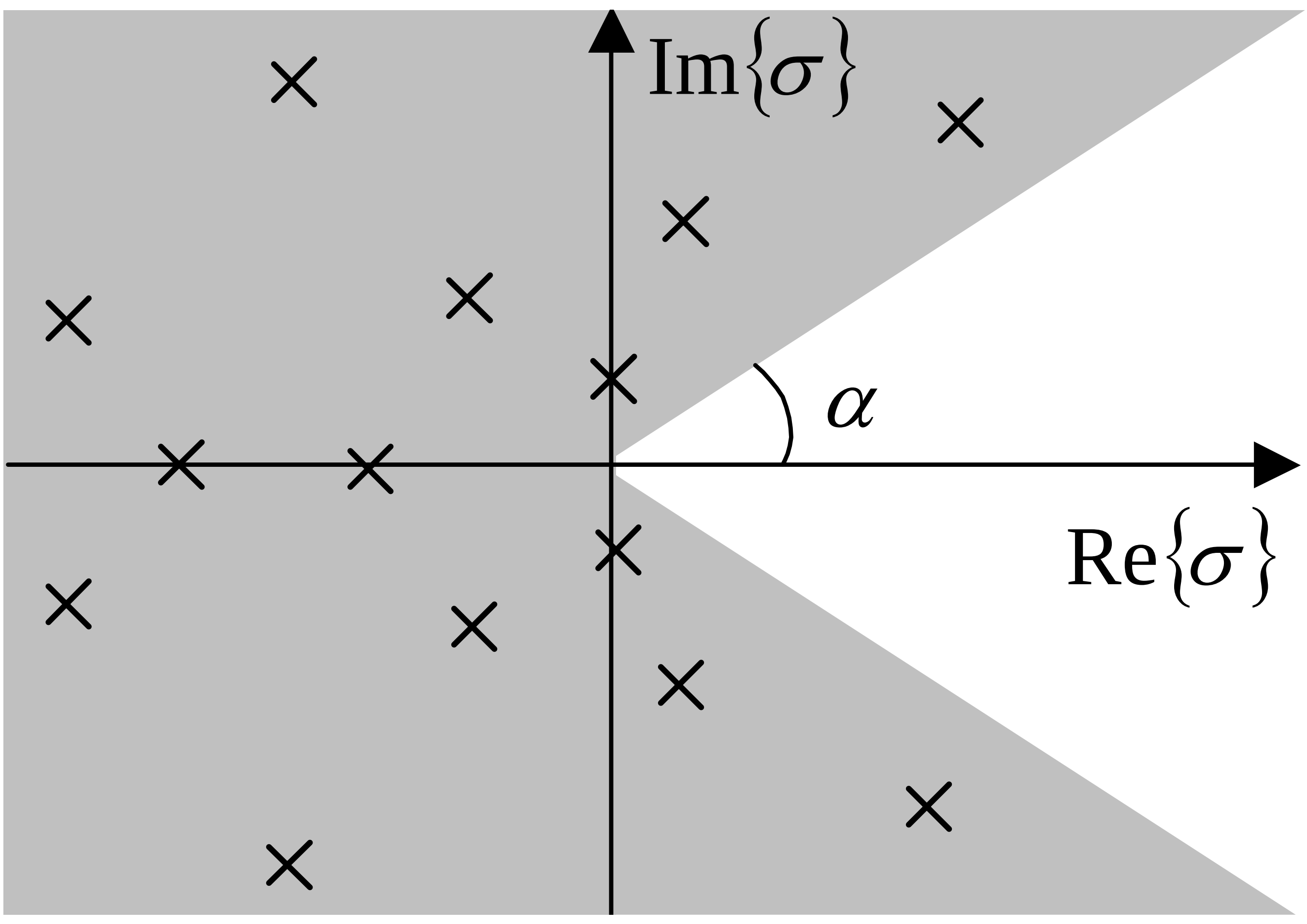

Fig. 2.14 Shadowed stability range for $0 < \nu < 1$

For $n = 1$

$$\nu = \frac{1}{n} = 1$$

$$\left|\arg\{\sigma_i\}\right| > \frac{\pi}{2} = \alpha$$

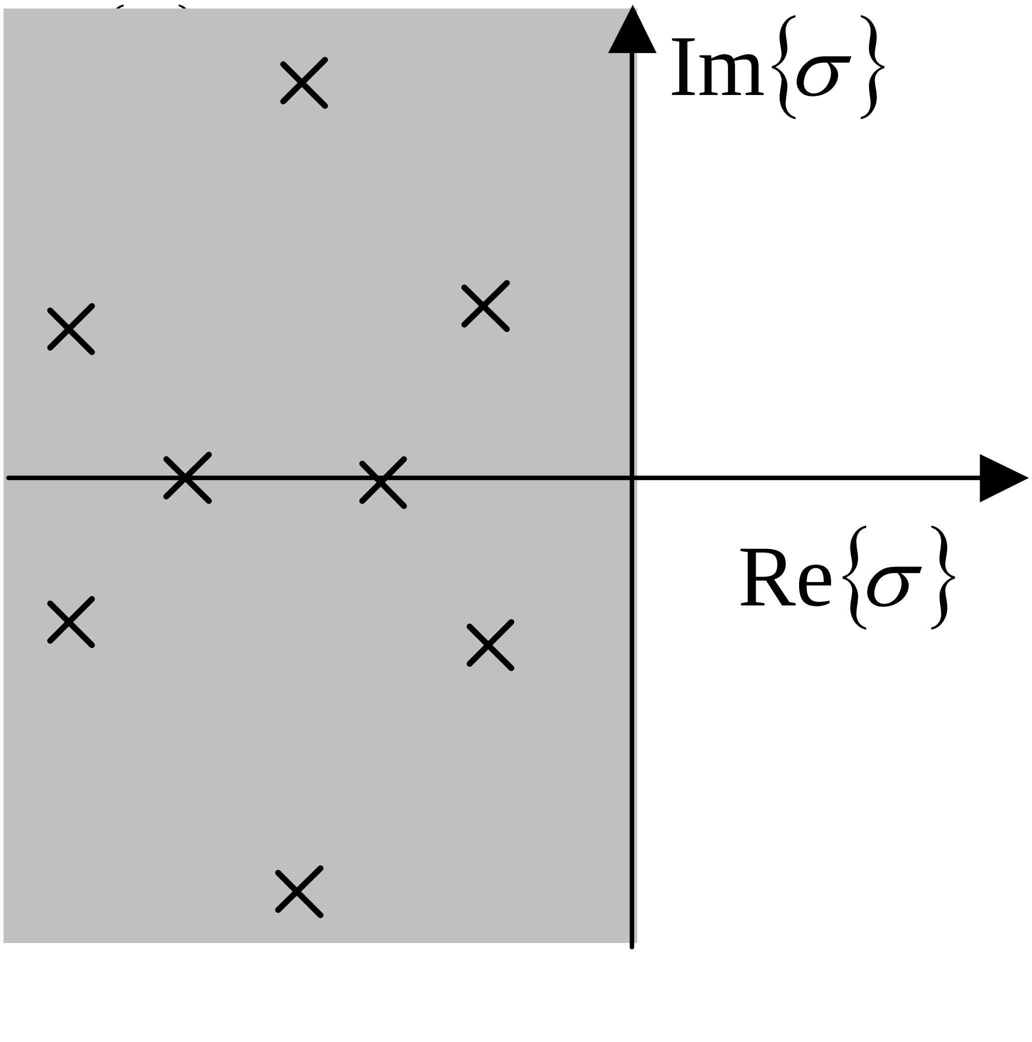

Fig. 2.15 Shadowed stability range for $\nu = 1$

2.6Identification of a fractional system

Identification of an ultracapacitor (gold capacitor)

1. High capacitance (≈ kF),
2. Low weight (in comparison to classical cells),
3. High charging/discharging time ,
4. Parameters constancy due to repeated charges and discharges (up to 10^6).
5. Decreasing capacitor voltage due to discharge time
6. Low maximal voltages (≈4V)

Fig. 2.16 Ultracapacitors

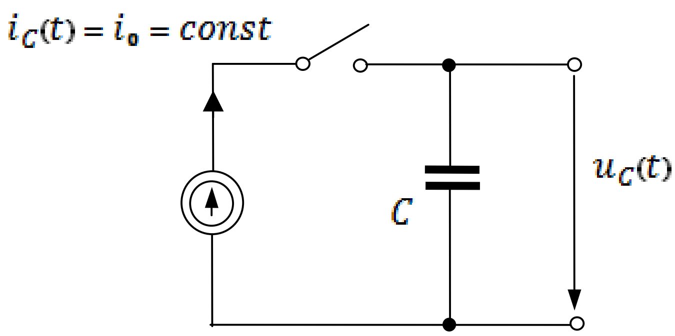

Fig. 2.17 Meter circuit

Mathematical model of the ultracapacitor (gold capacitor)

$$C\,{}^{GL}_{\;0}D_t^{(\nu)}u_C(t) = i_C(t)$$

Fractional transfer function

$$G(s) = \frac{U_C(s)}{I_C(s)} = \frac{1}{Cs^{\nu}}$$

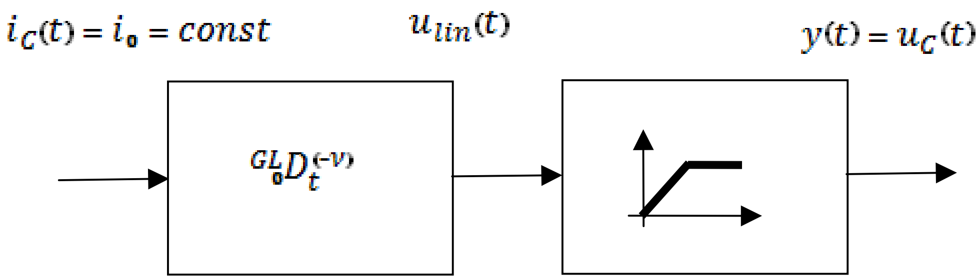

Fig. 2.17 Block diagram of the Wiener model of the ultracapacitor

$$C = 1[F],$$

$$\nu = 0.341$$

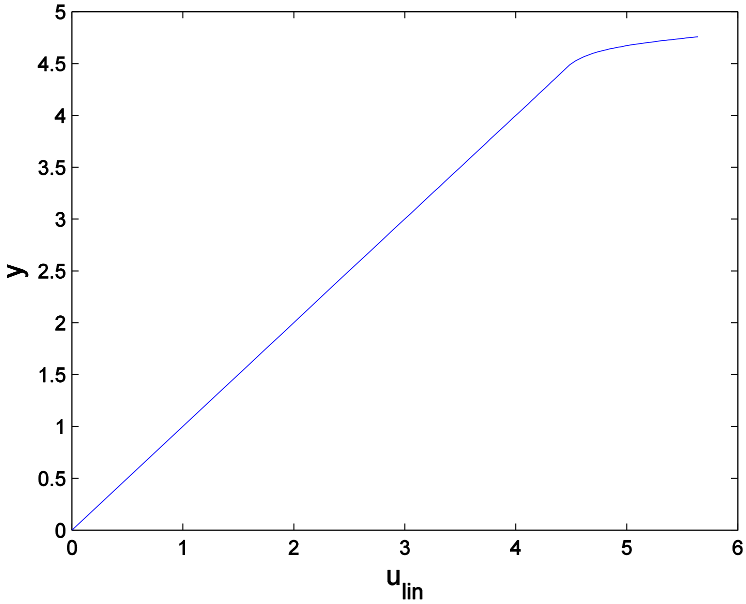

Fig. 2.18 Statical characteristic

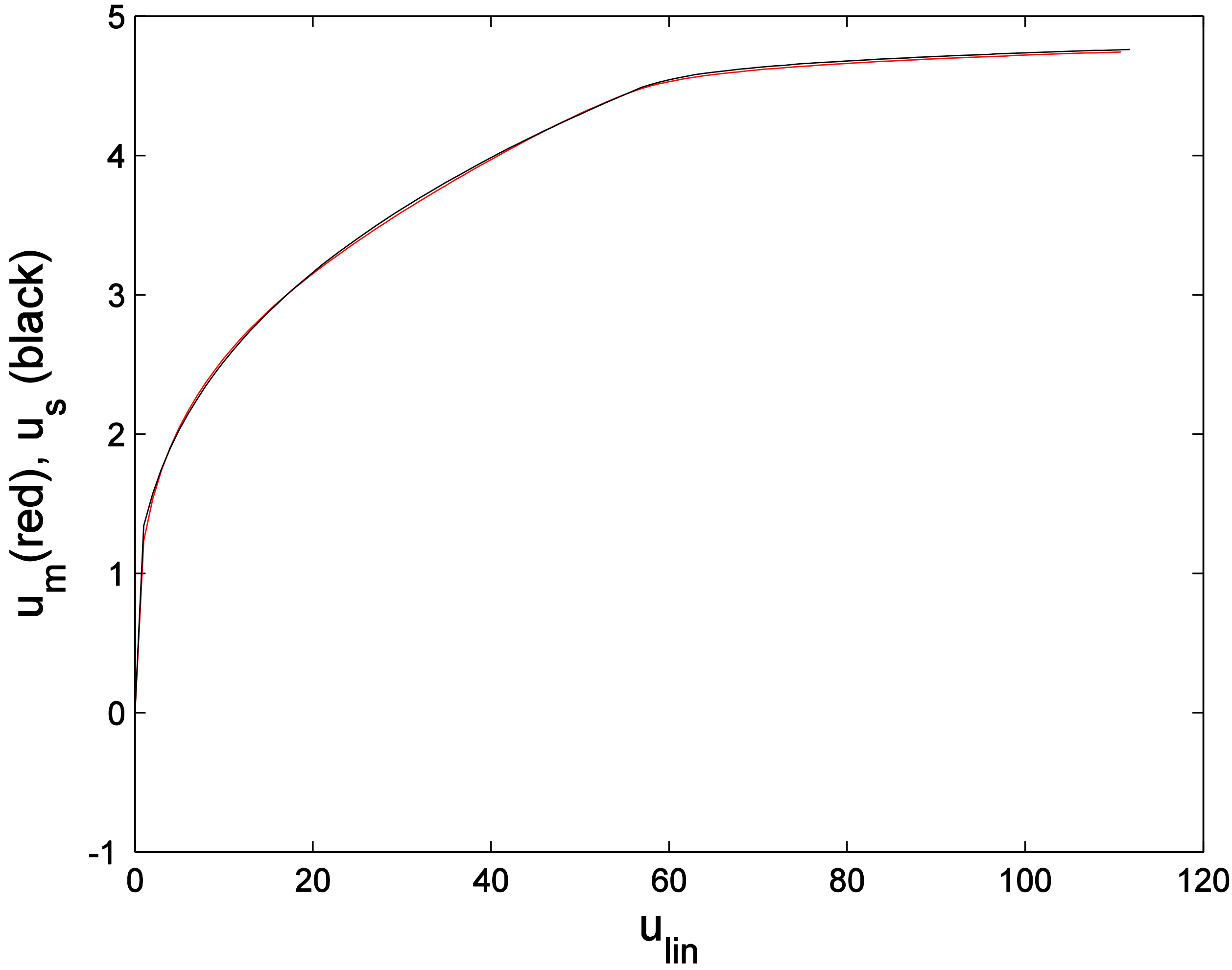

Fig. 2.19 Charging characteristics: measured (red) and simulated (black)

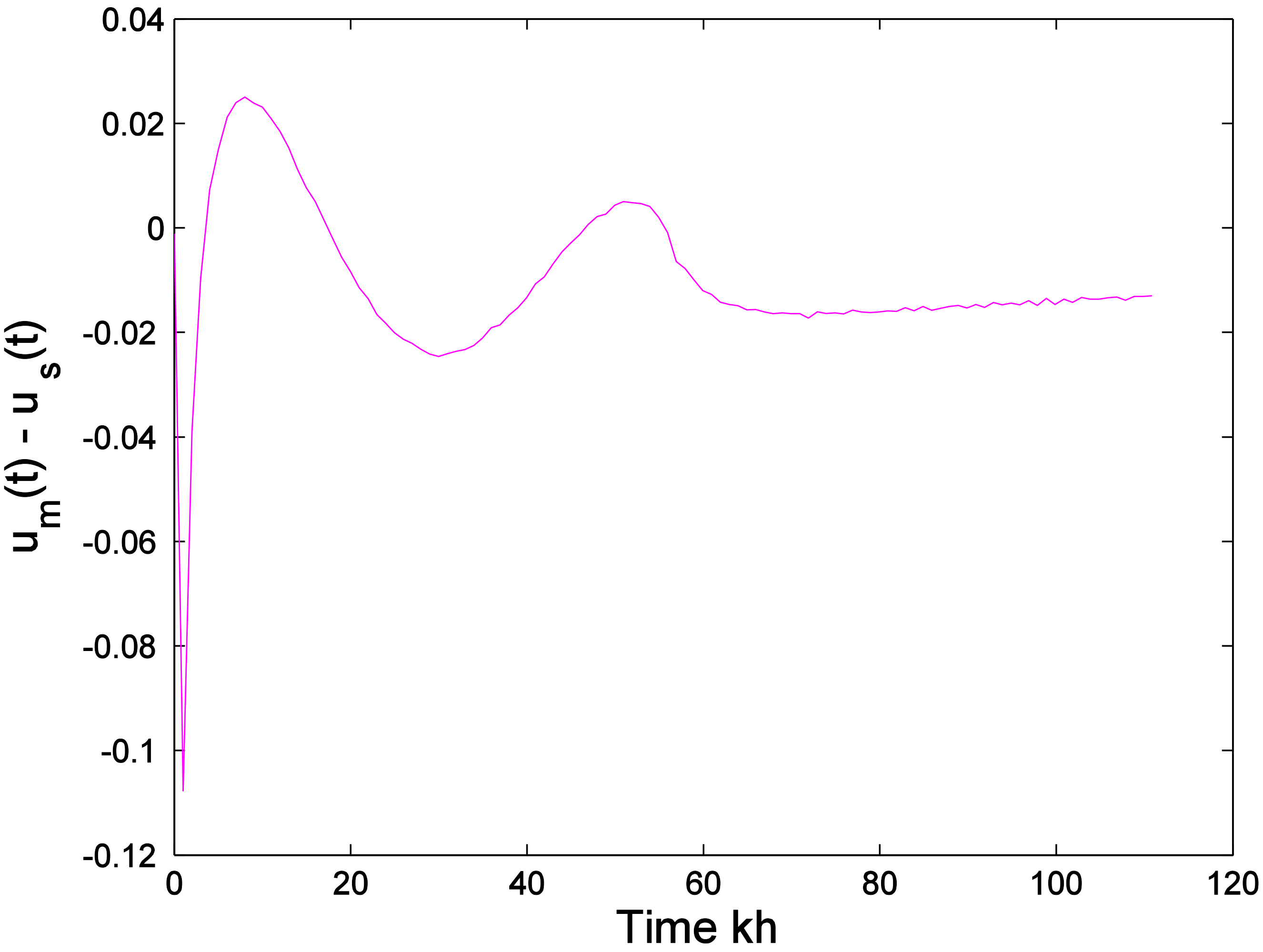

Fig. 2.20 Error between measured and simulated transient characteristics [V]

2.7 Fractional control

2.7.1 CRONE controller

Consider an open – loop system consisting of the linear, time – invariant plant described by the classical transfer function $G_o(s,\mathbf{p})$

$$G_o(s,\mathbf{p}) = \frac{Y(s)}{V(s)} =$$

$$\frac{b_m(\mathbf{p})s^m + b_{m-1}(\mathbf{p})s^{m-1} + \cdots + b_1(\mathbf{p})s^1 + b_0(\mathbf{p})}{s^n + a_{n-1}(\mathbf{p})s^n + \cdots + a_1(\mathbf{p})s^1 + a_0(\mathbf{p})}$$

where $\mathbf{p}$ – plant parameter vector

$$\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_r \end{bmatrix}, \quad p_i \in \left[p_{l,i}, p_{h,i}\right], \text{ for } i = 1,2,\cdots,r$$

and classical controller (compensator)

$$C(s) = \frac{b_{C,m_C} s^{m_C} + b_{C,m_C-1} s^{m_C-1} + \cdots + b_{C,1} s^1 + b_{C,0}}{s^{n_C} + a_{C,n_C-1} s^{n_C-1} + \cdots + a_{C,1} s^1 + a_{C,0}}$$

Relation between an open - and closed – loop characteristics FIs and classical systems

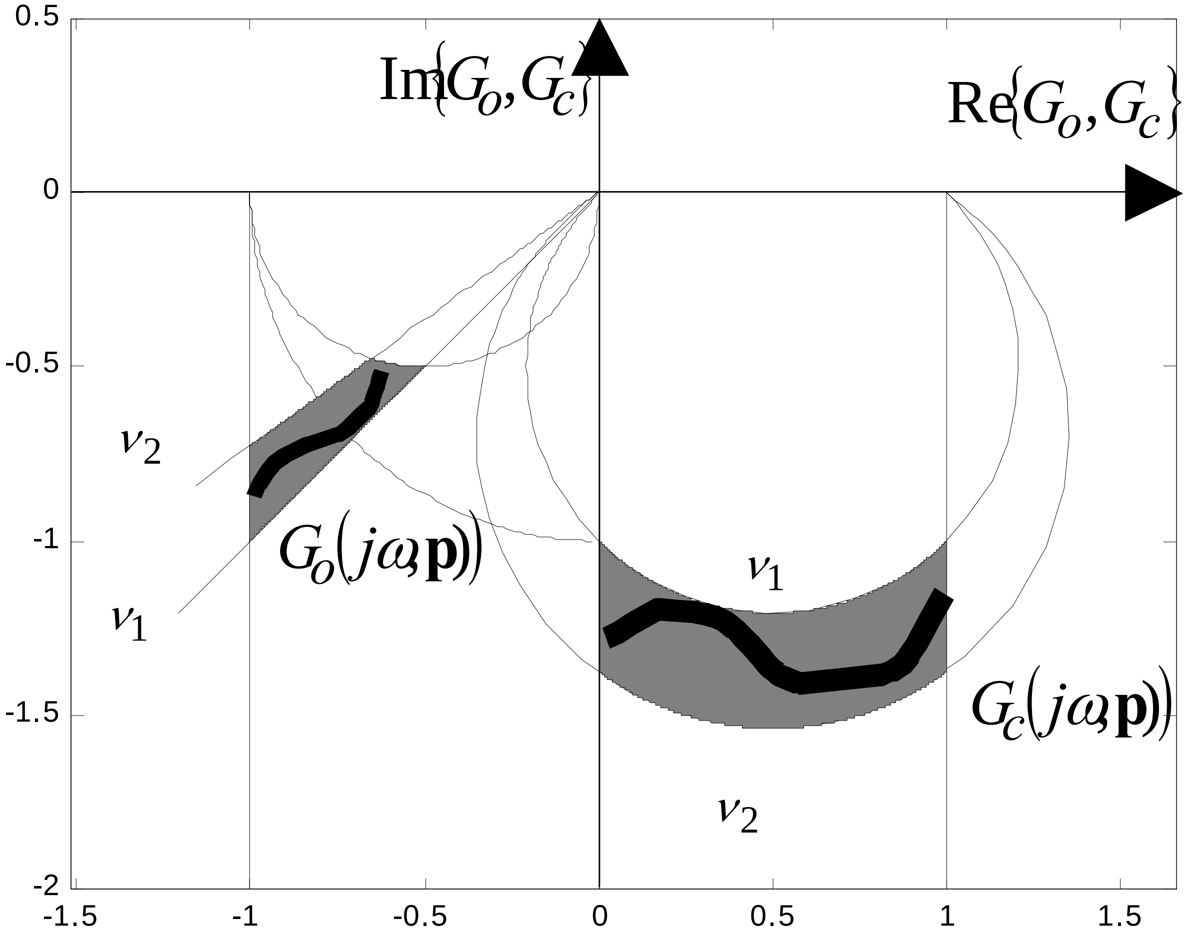

Fig. 2.21 Relation between an open - and closed – loop characteristics

The CRONE controller idea (fr. **C**ommande **R**obuste d'**O**rdre **N**on **E**ntiere)

$$G_O(s,\mathbf{p}_n)C_P(s)C(s)\Big|_{s=j\omega} = \frac{1}{(sT_u)^{\mu}}\Bigg|_{s=j\omega}$$

$$\omega \in \left[\omega_u - \omega_d, \omega_u + \omega_g\right]$$

$$\omega_d, \omega_g, \omega_u = \frac{1}{T_u} > 0$$

Example 2.4

$$G_O(j\omega,\mathbf{p}) = \frac{p_1}{(sp_2+1)(sT_1+1)}$$

$$\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \text{ for } \begin{matrix} p_1 \in [2.5 \quad 10] \\ p_2 \in [0.9 \quad 1.1] \end{matrix}.$$

$$T_1 = 1.$$

One assumes a pre-compensator

$$C_P(s) = K_P \frac{1 + 0.3s}{sT_I}$$

$K_P = 1, T_I = 1$

$$\nu = \frac{23}{18}$$

$$\omega_u = 3.1692$$

$$C(s) = k_c \frac{\prod_{i=1}^{6} (s - z_{ci})}{\prod_{i=1}^{6} (s - s_{ci})}$$

with

$k_c = 3.6178e + 01$

$$z_{c1} = -2.5291e+01$$

$$z_{c2} = -7.6659e+00$$

$$z_{c3} = -3.5059e+00$$

$$z_{c4} = -1.8182e+00$$

$$z_{c5} = -1.0541e+00 + j1.4281e-01$$

$$z_{c6} = -1.0541e+00 - j1.4281e-01$$

$$z_{c7} = -0.3000e00$$

$$s_{c1} = -2.2068\mathrm{e}+02$$

$$s_{c2} = -1.6652\mathrm{e}+01$$

$$s_{c3} = -6.0526\mathrm{e}+00$$

$$\mathrm{s}_{\mathrm{c}4} = -2.8832e+00$$

$$s_{c5} = -1.5523e+00$$

$$s_{c6} = -1.5045e-01$$

$$s_{c7} = 0$$

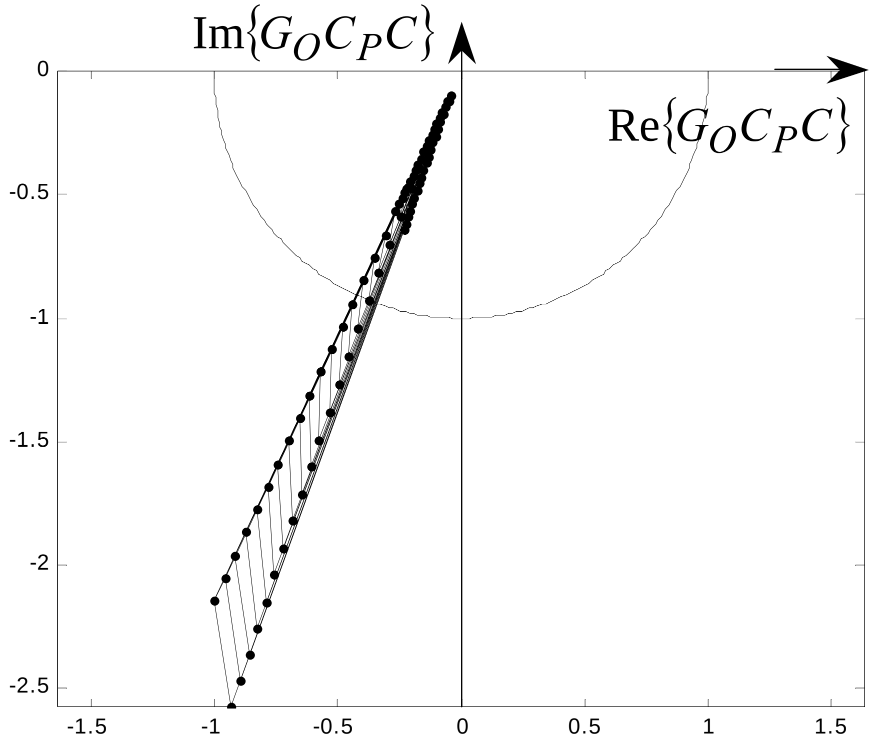

Fig. 2.22 Nyquist plot of the open – loop system with simplified uncertainty range

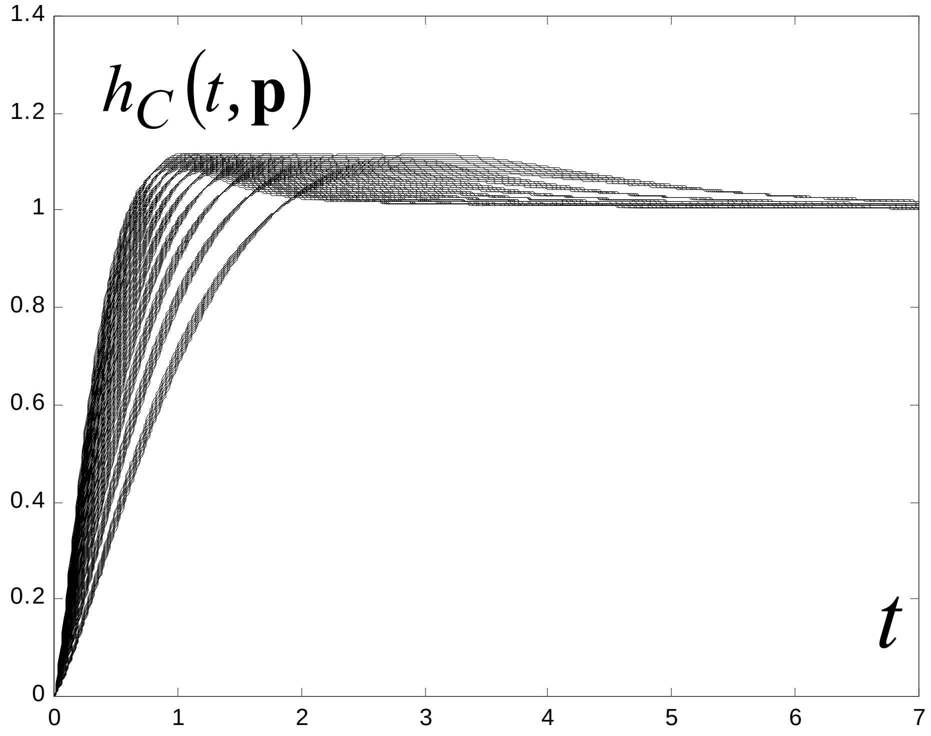

Fig. 2.23 Step responses of the closed – loop system with CRONE controller for $p_1 = \text{var}$ and $p_2 = \text{var}$.

2.7.2 FO PID controller

$$v(t) = K_p\left[e(t) + \frac{1}{T_I}\,{}_0I_t^{(\nu)}e(t) + T_D\,{}_0D_t^{(\mu)}e(t)\right]$$

K_P, T_I, T_D - controller parameters,

ν - integration order,

μ - differentiation order.

$$R_{PI^{(\nu)}D^{(\mu)}}(s) = \frac{V(s)}{E(s)} = K_P\left[1 + \frac{1}{T_I s^{\nu}} + T_D s^{\mu}\right]$$

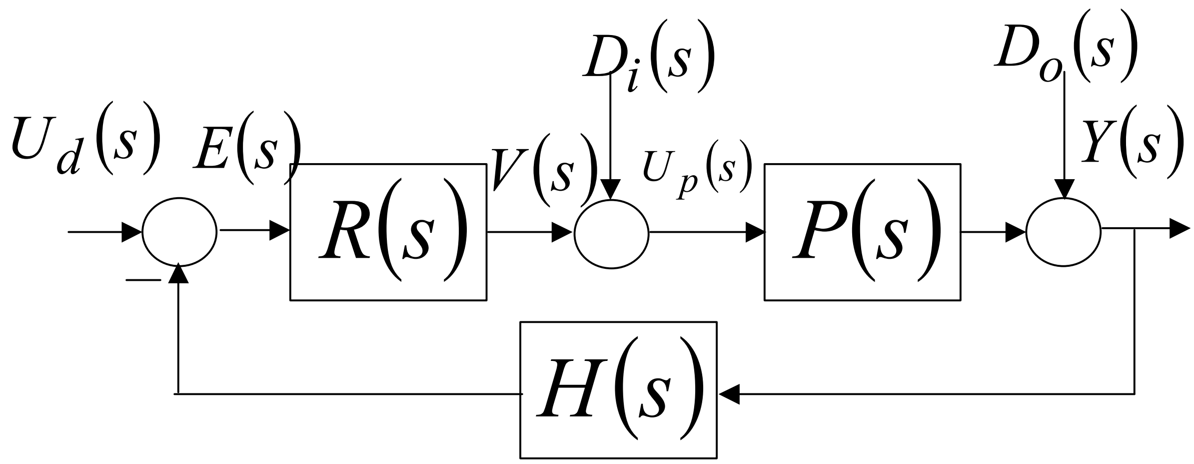

Fig.2.24 Block diagram of a closed – loop system with FO controller

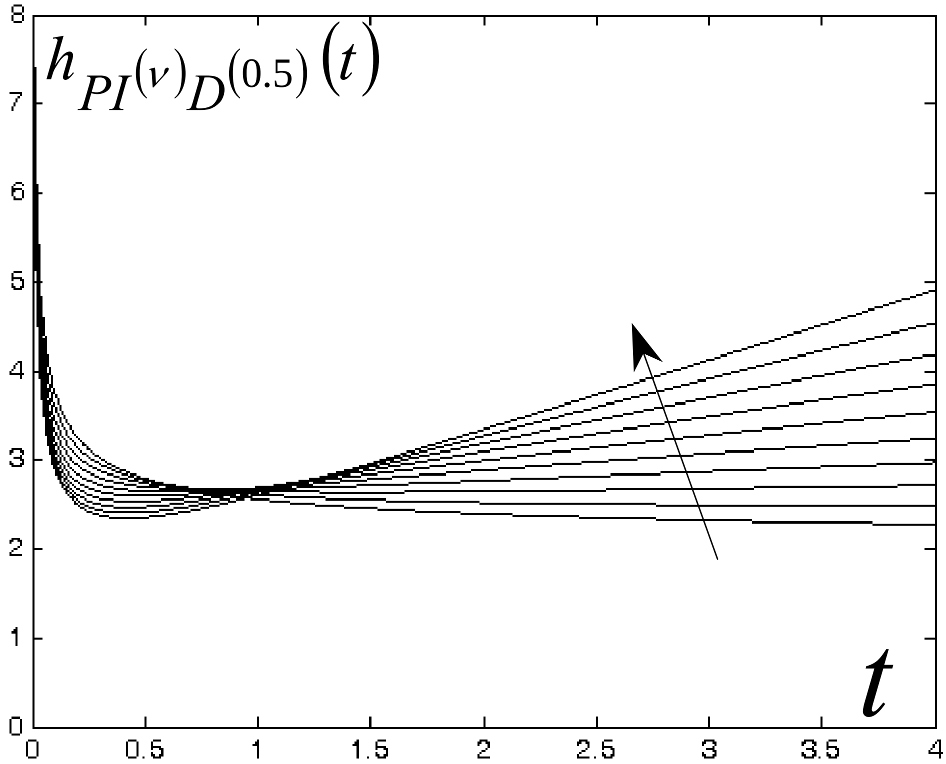

Fig.2.25 Step responses of the ideal FO PID controller $PI^{(\nu)}D^{(0.5)}$ for $\nu \in \{0.0, 0.1, \cdots, 0.9\}$

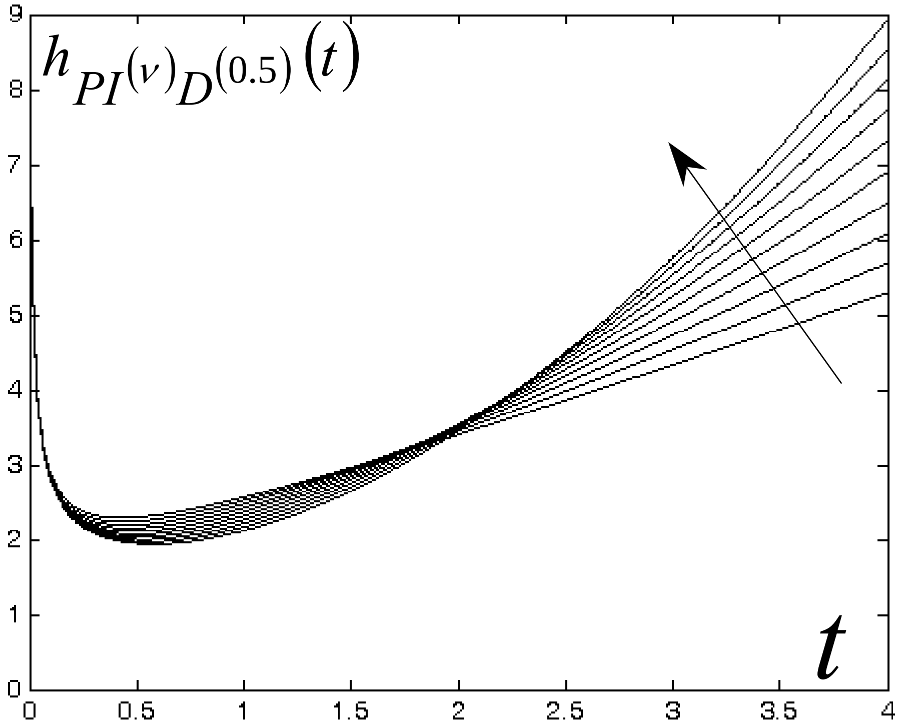

Fig.2.26 Step responses of the ideal FO PID controller $PI^{(\nu)}D^{(0.5)}$ for $\nu \in \{1.0, 1.1, \cdots, 1.9\}$

Nyquist characteristics of the FO PID controller

$$\left[\left[\sin\left(\mu\frac{\pi}{2}\right) \quad -\cos\left(\mu\frac{\pi}{2}\right)\right]\begin{bmatrix}\frac{P}{K_P}-1\\ \frac{Q}{K_P}\end{bmatrix}\right]^{\frac{1}{\nu}}\left[\left[\sin\left(\nu\frac{\pi}{2}\right) \quad \cos\left(\nu\frac{\pi}{2}\right)\right]\begin{bmatrix}\frac{P}{K_P}-1\\ \frac{Q}{K_P}\end{bmatrix}\right]^{\frac{1}{\mu}} =$$

$$\left[\frac{\sin\left[(\nu+\mu)\frac{\pi}{2}\right]}{T_I}\right]^{\frac{1}{\nu}}\left[T_D\sin\left[(\nu+\mu)\frac{\pi}{2}\right]\right]^{\frac{1}{\mu}}$$

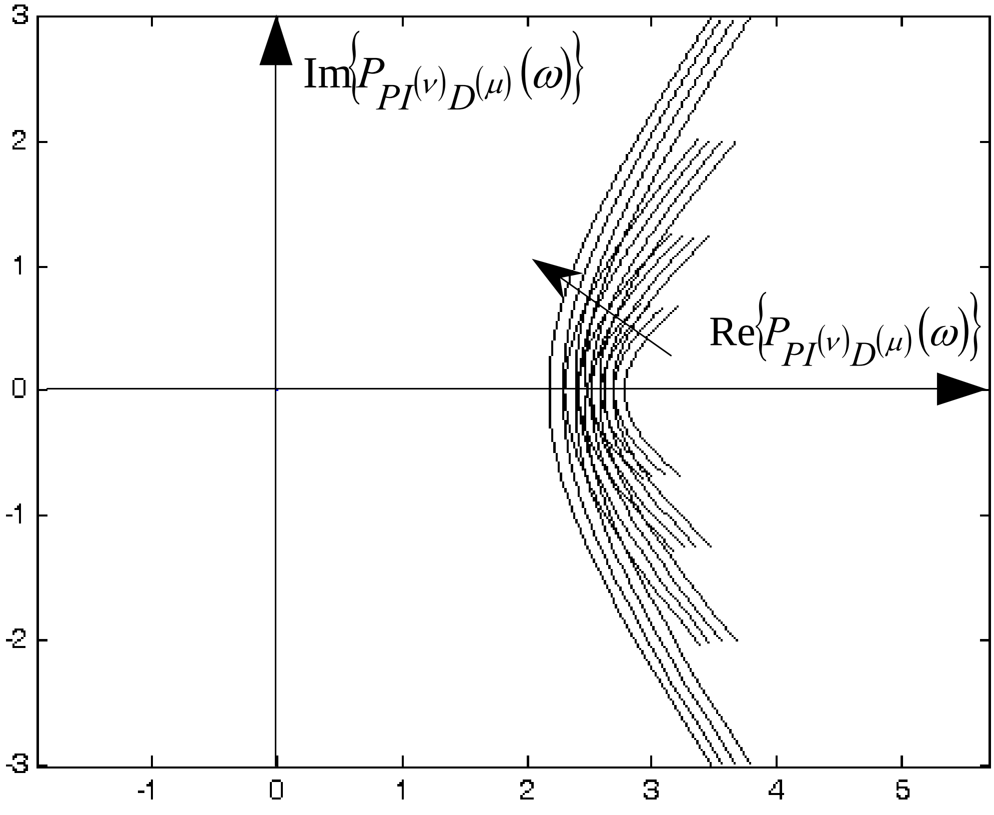

Fig.2.27 Nyquist characteristics of the $PI^{(\nu)}D^{(\mu)}$ for $K_P = T_I = T_D = 1$
$\mu = \nu \in \{0.3 \quad 0.4 \quad 0.5 \quad 0.6\}$