FRACTIONAL CALCULUS IN AUTOMATICS

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Contens

1. FOD/I

Consider a set of n-fold integrals and n-order derivatives

$$\ldots,\ \int_{t_0}^{t}\left[\int_{t_0}^{\tau_1}\left[\int_{t_0}^{\tau_2} f(\tau_3)d\tau_3\right]d\tau_2\right]d\tau_1,$$

$$\int_{t_0}^{t}\left[\int_{t_0}^{\tau_1} f(\tau_2)d\tau_2\right]d\tau_1, \qquad \int_{t_0}^{t} f(\tau)d\tau,$$

$$f(t),$$

$$\frac{df(t)}{dt}, \qquad \frac{d^2 f(t)}{dt^2},$$

$$\frac{d^3 f(t)}{dt^3}, \ldots$$

Uniform notation (Davies, 1927)

$$_{t_0}D_t^{(n)} f(t)$$

t_0, t – terminals of a integration

$n \in \mathbf{Z}$ - integration/differentiation order

$$ {}_{t_0}D_t^{(n)} f(t) = $$

$$ \begin{cases} \dfrac{d^n f(t)}{dt^n} & \text{for} \quad n > 0 \\ f(t) & \text{for} \quad n = 0 \\ \int_{t_0}^{t} \left[\int_{t_0}^{\tau_1} \cdots \left[\int_{t_0}^{\tau_{-n-1}} f(\tau_{-n}) d\tau_{-n} \right] \cdots d\tau_2 \right] d\tau_1 & \text{for} \quad n < 0 \end{cases} $$

$$ {}_{t_0}D_t^{(n)} f(t) = {}_{t_0}I_t^{(-n)} f(t) $$

$$ {}_{t_0}D_t^{(-n)} f(t) = {}_{t_0}I_t^{(n)} f(t) $$

D - derivative
I - integral

$$ n, m \in \mathbf{Z} $$

$$ \nu, \mu \in \mathbf{R} / \mathbf{Z} $$

Derivatives and integrals of positive orders can be treated as an approximation of the series of classical integer – order operators.

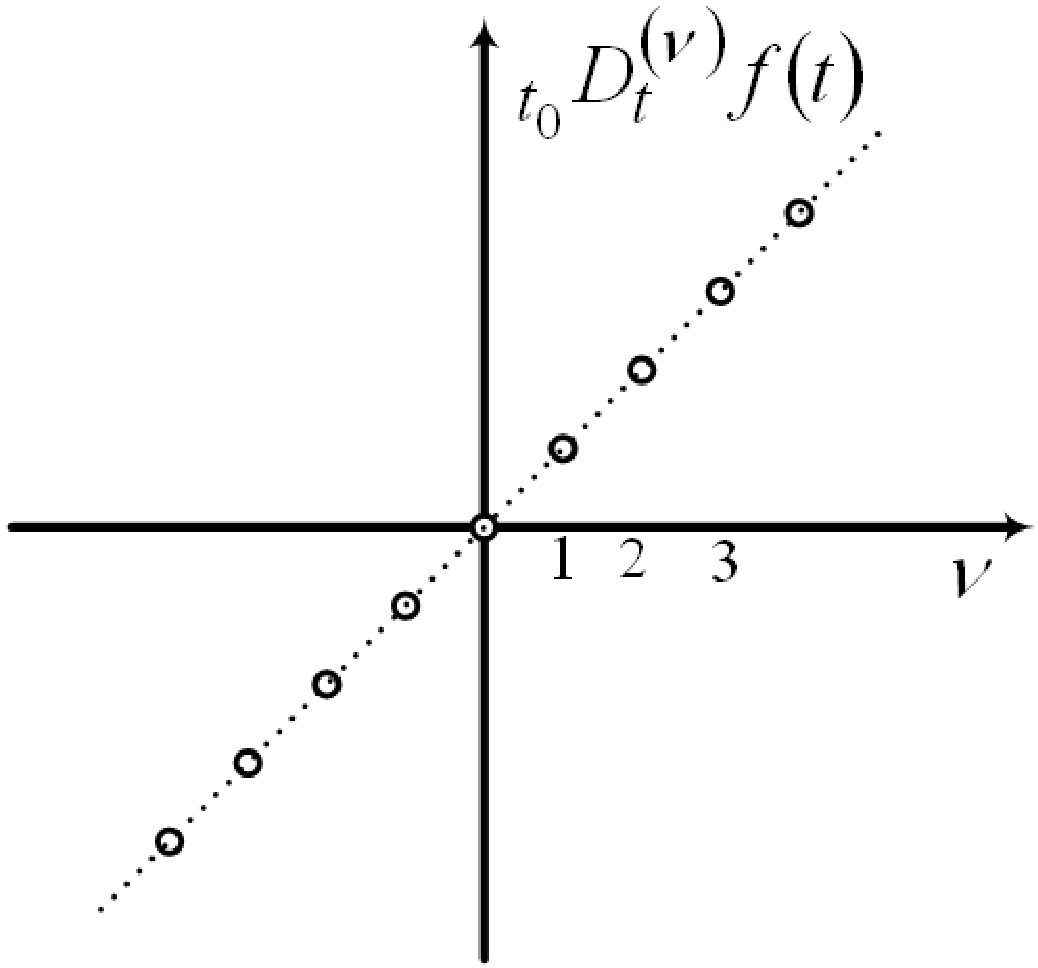


Fig.1.1 Order continuity of differentiation and integration operators

1.1 Grünwald – Letnikov fractional FOD/I

Consider a real, bounded function $f(t)$ defined on $[t_0, t]$ with

$$h = \frac{t - t_0}{k}$$

For $t - t_0 = const$, $k \to +\infty$ and $h \to 0$

$${}_{t_0}D_t^{(\nu)} f(t) = {}_{t_0}I_t^{(-\nu)} f(t)$$

$${}_{t_0}D_t^{(-\nu)} f(t) = {}_{t_0}I_t^{(\nu)} f(t)$$

Definition 1.1

Grünwald – Letnikov left- sided differ-integral of order ν of a real function is defined by a limit of a sum

$${}_{t_0}D_t^{(\nu)} f(t) = \lim_{\substack{h \to 0 \\ t - t_0 = kh}} \left[\frac{1}{h^{\nu}} \sum_{i=0}^{k} a_i^{(\nu)} f(t - hi) \right]$$

or

$$ {}_{t_0}D_t^{(\nu)} f(t) = $$

$$ \lim_{\substack{h\to 0\\ t-t_0=kh}} \frac{1}{h^\nu} \begin{bmatrix} a_0^{(\nu)} & a_1^{(\nu)} & \cdots & a_k^{(\nu)} \end{bmatrix} \begin{bmatrix} f(t) \\ f(t-h) \\ \vdots \\ f(t-kh) \end{bmatrix} $$

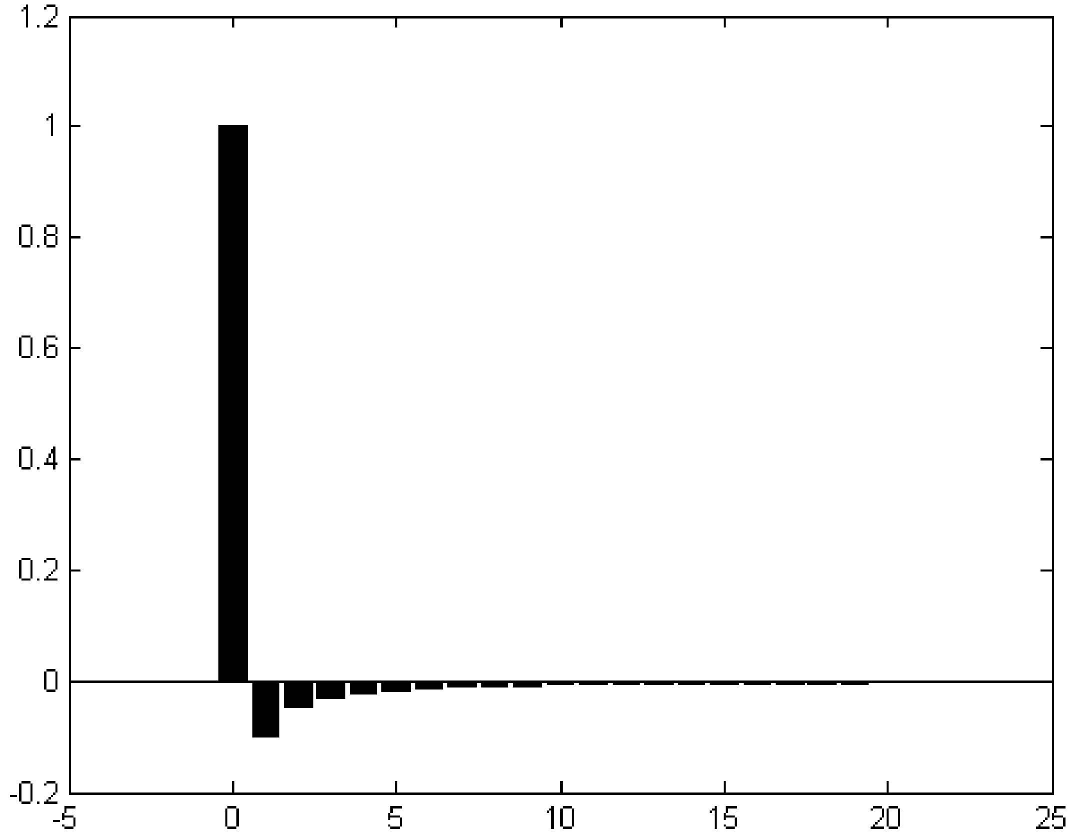
where coefficients $a_i^{(\nu)}$ are defined as

$$ a_i^{(\nu)} = \begin{cases} 1 & \text{for} \quad i = 0 \\ (-1)^i \dfrac{\nu(\nu-1)(\nu-2)\cdots(\nu-i+1)}{i!} & \text{for} \quad i = 1,2,3,\cdots \end{cases} $$

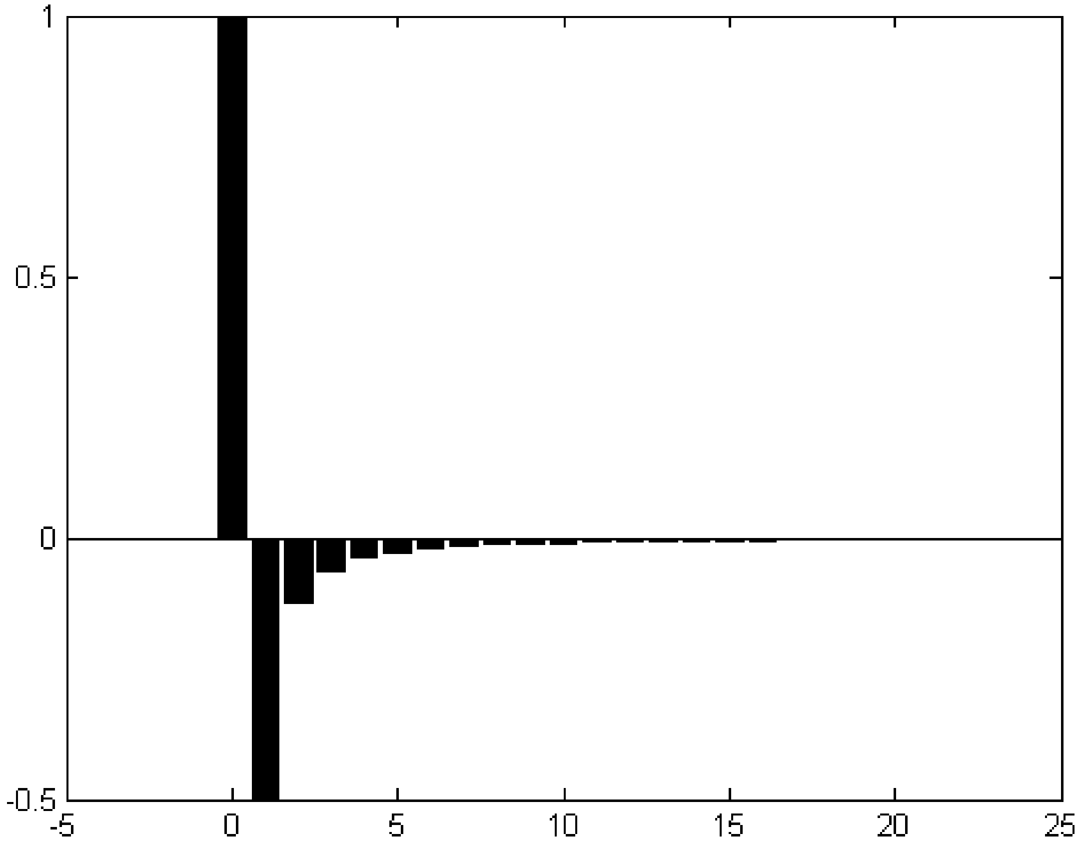
or

$$ a_i^{(\nu)} = a_{i-1}^{(\nu)} \left(1 - \frac{\nu+1}{i} \right), $$

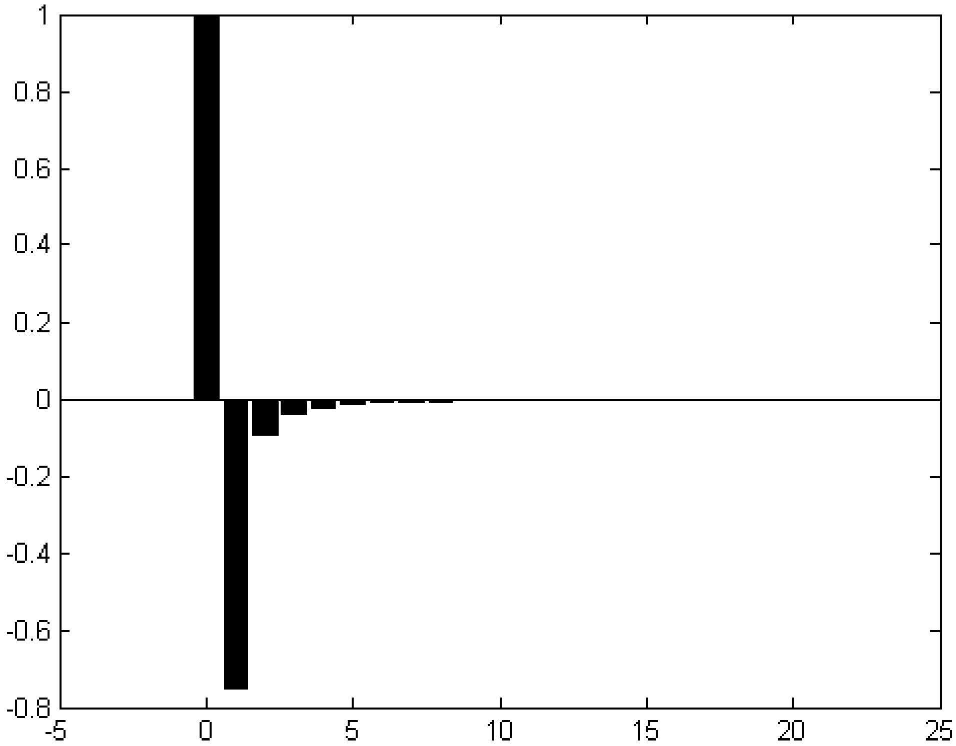
$$ a_0^{(\nu)} = 1 $$



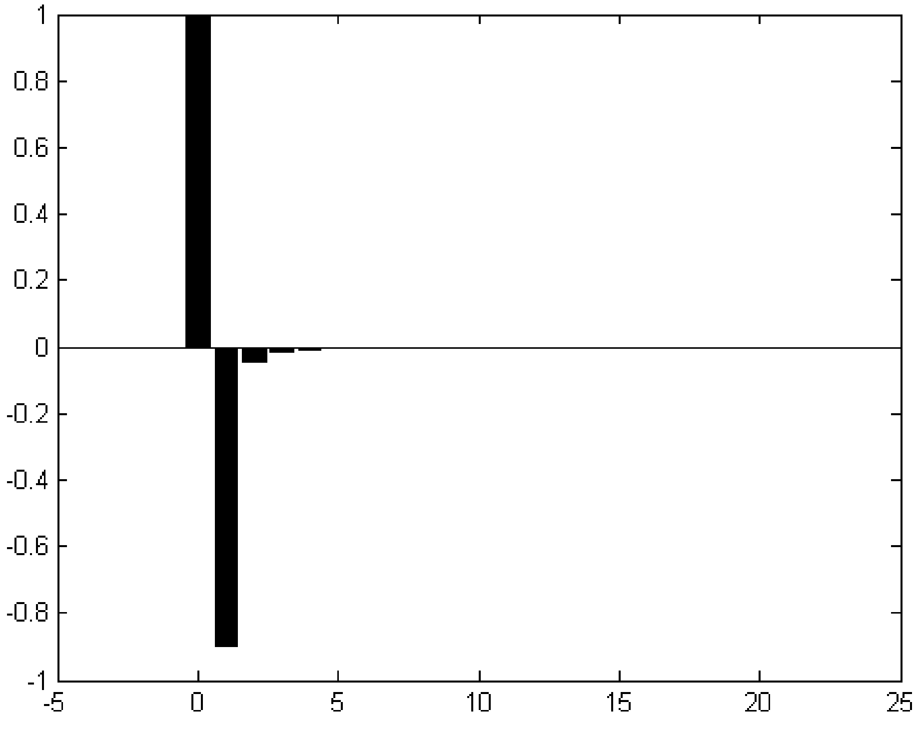
$$\nu = 0.1$$



$$\nu = 0.5$$

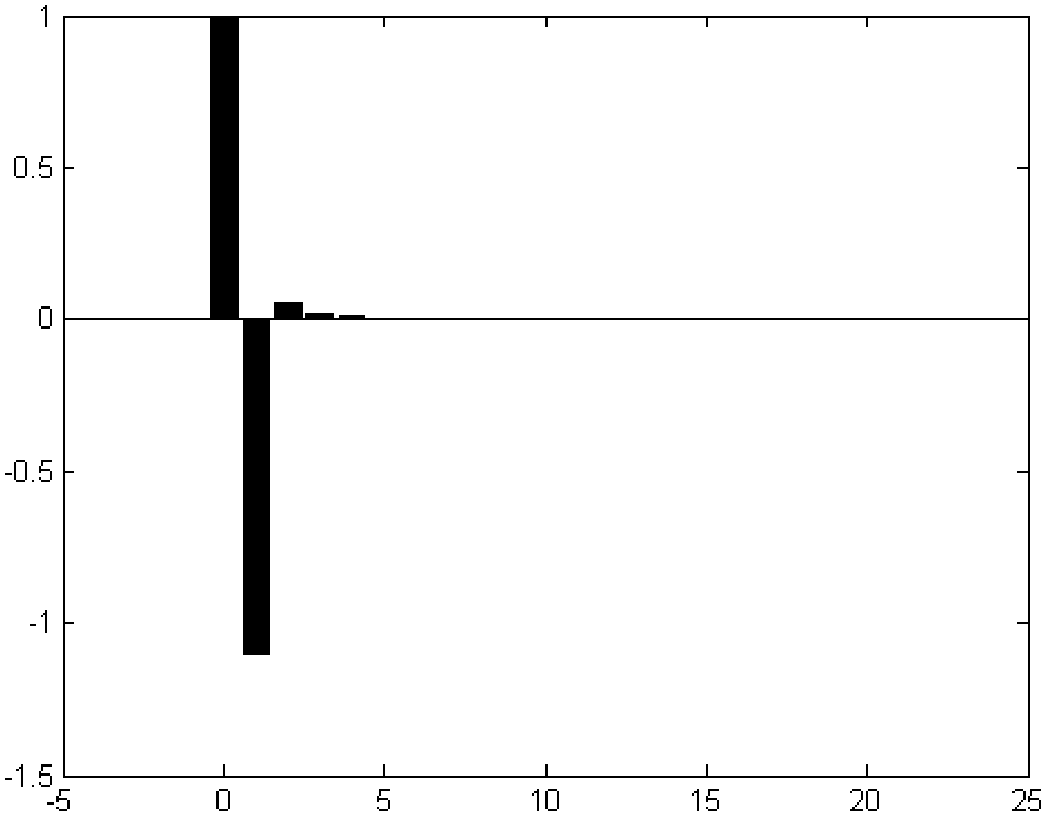


$\nu = 0.75$

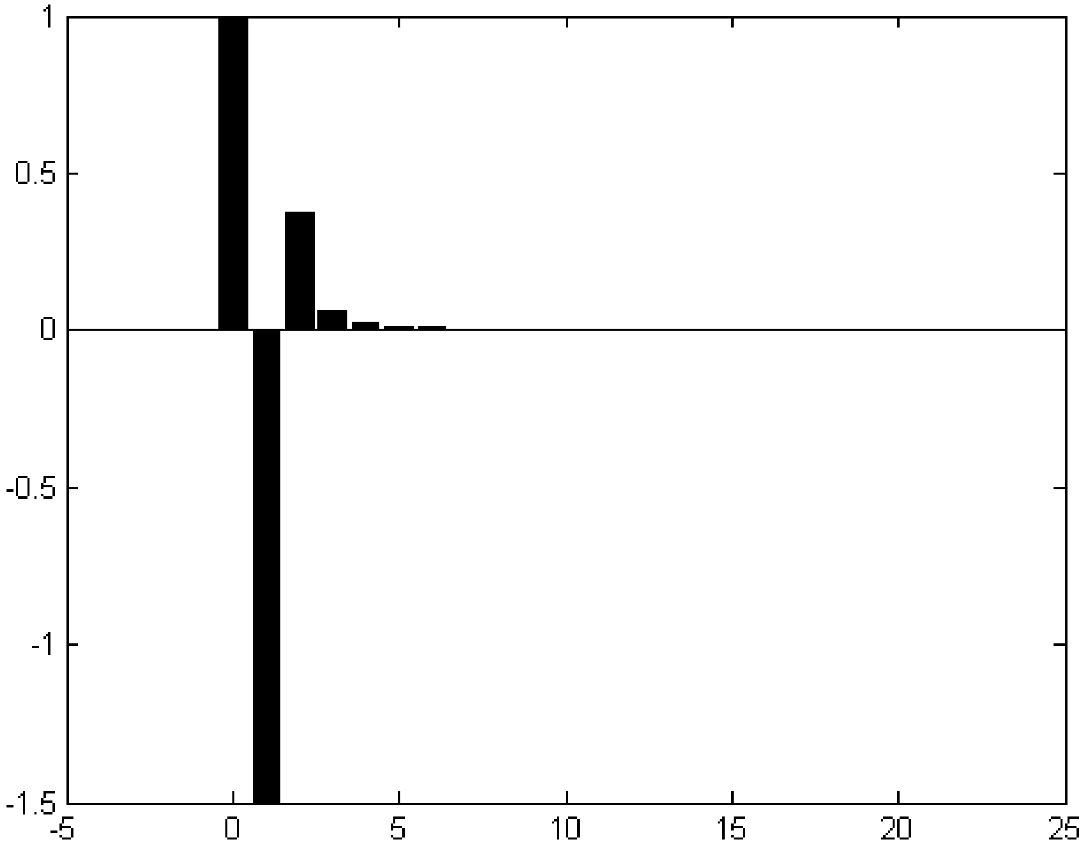


$\nu = 0.9$

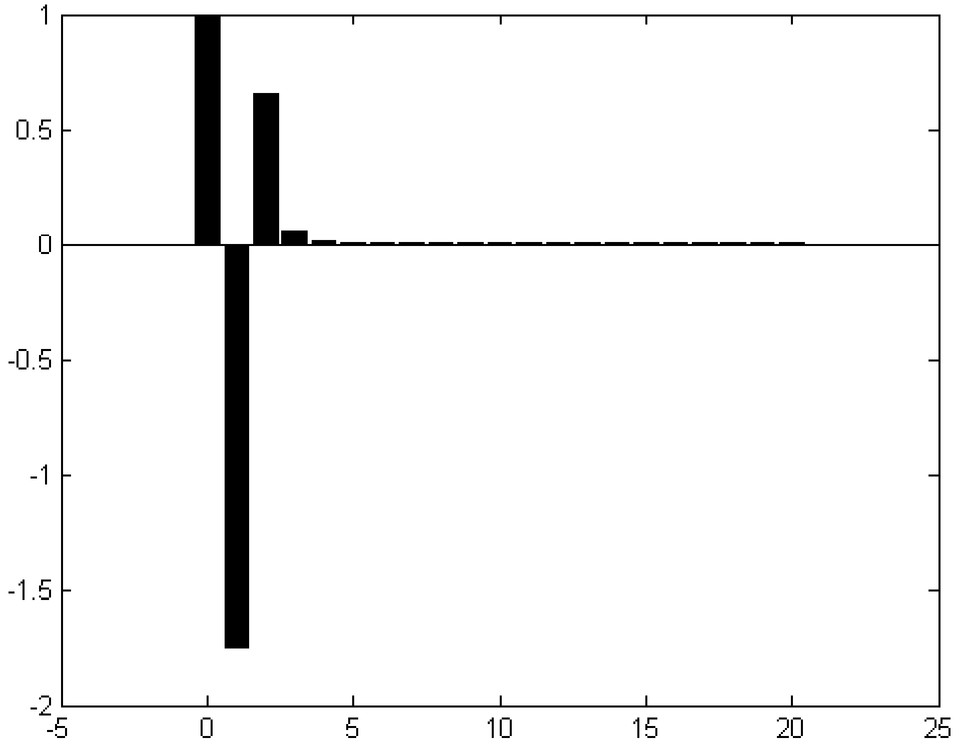
Fig.1.2 Coefficients $a_j^{(\nu)}$ vs. j for $\nu = const$ from $(0,1)$



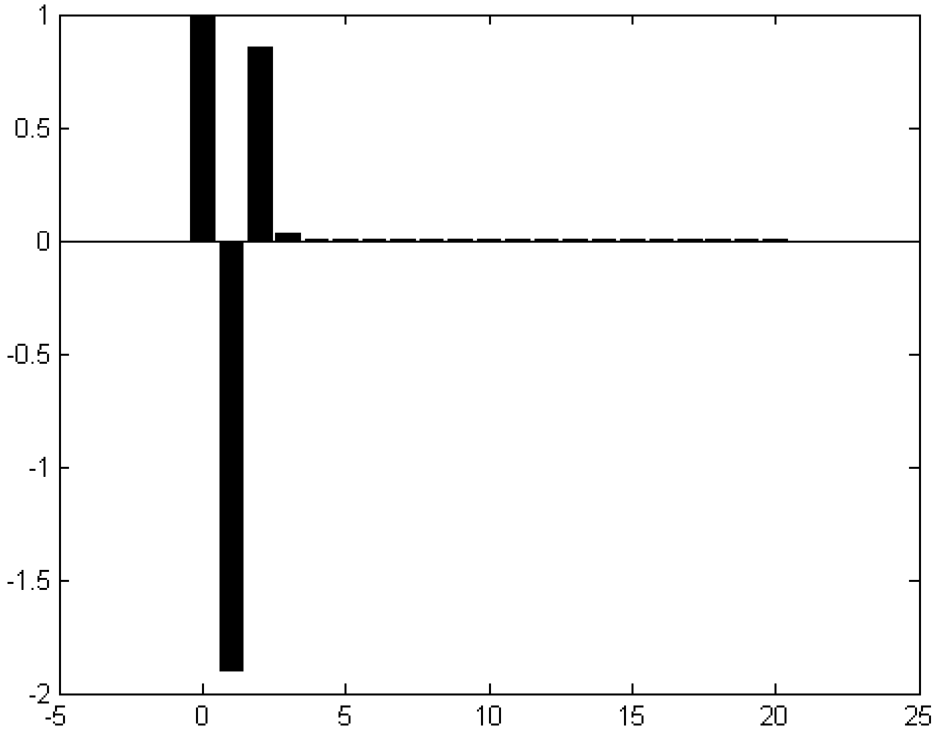
$\nu = 1.1$



$\nu = 1.5$



$\nu = 1.75$



$\nu = 1.9$

Fig.1.3 Coefficients $a_j^{(\nu)}$ vs. j for $\nu = const$ from $(1,2)$.

Example 1.1

Evaluate the fractional – order $\nu \in \mathrm{R}_+$ left – sided derivative of a function $f(t) = \mathbf{I}(t)$ on $(0, t]$

$$ {}_0D_t^{(\nu)}\mathbf{I}(t) = $$

$$ \lim_{k\to\infty}\left[\frac{1}{\left(\frac{t}{k}\right)^{\nu}} \sum_{i=0}^{k} a_i^{(\nu)} \mathbf{I}\left(t - \frac{ti}{k}\right)\right] = \lim_{k\to\infty}\left[\left(\frac{t}{k}\right)^{-\nu} \sum_{i=0}^{k} a_i^{(\nu)}\right] = $$

$$ \lim_{k\to\infty}\left[t^{-\nu} \sum_{i=0}^{k} k^{\nu} a_i^{(\nu)}\right] = $$

$$ t^{-\nu} \lim_{k\to\infty}\left(\sum_{i=0}^{k} k^{\nu} a_i^{(n)}\right) = \frac{t^{-\nu}}{\Gamma(1-\nu)} $$

$\Gamma(x)$- the Euler Gamma function.

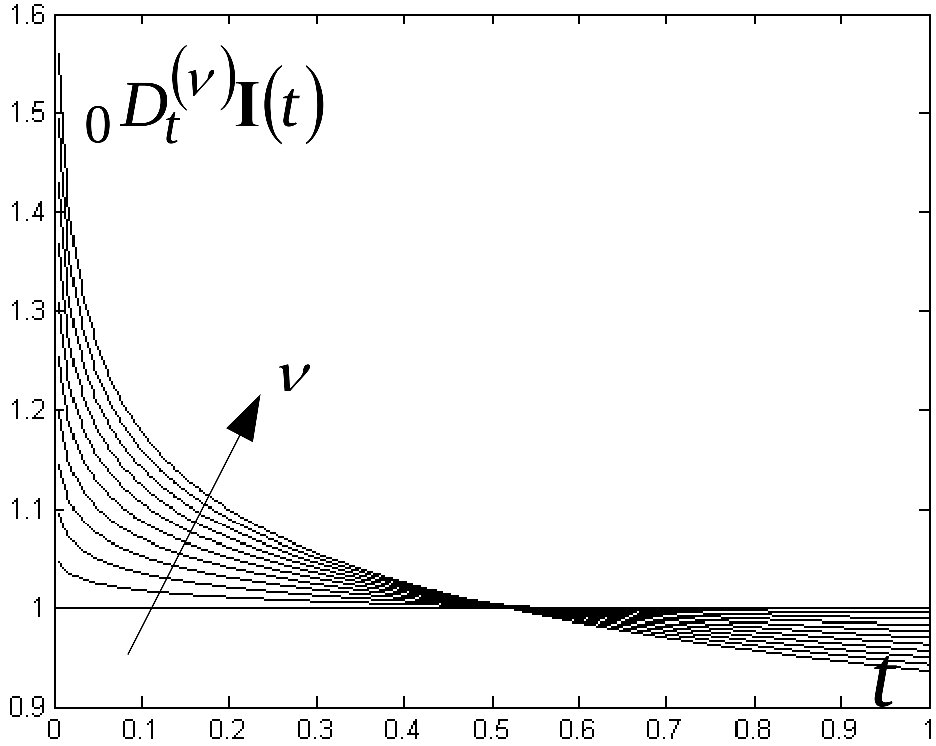


Fig.1.4 FOD of $f(t)=\mathbf{I}(t)$ of orders from a set $\{0.0, \quad 0.01, \quad \cdots \quad 0.09, \quad 0.1\}$.

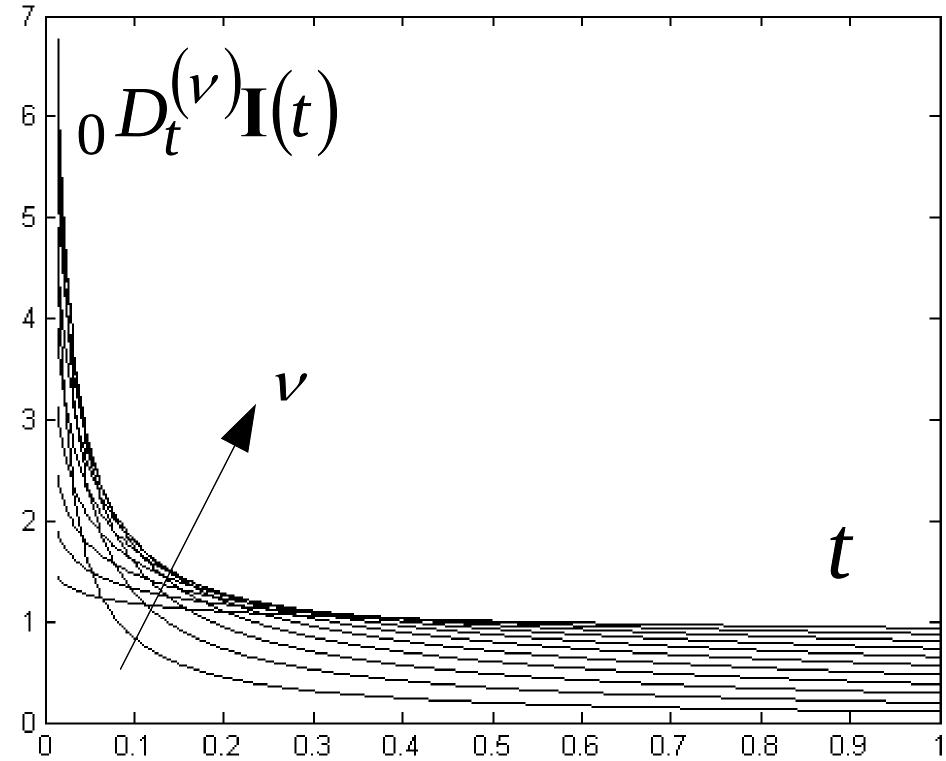


Fig.1.5 FOD of $f(t)=\mathbf{I}(t)$ of orders from a set. $\{0.1, \quad 0.2, \quad \cdots \quad 0.8, \quad 0.9\}$.

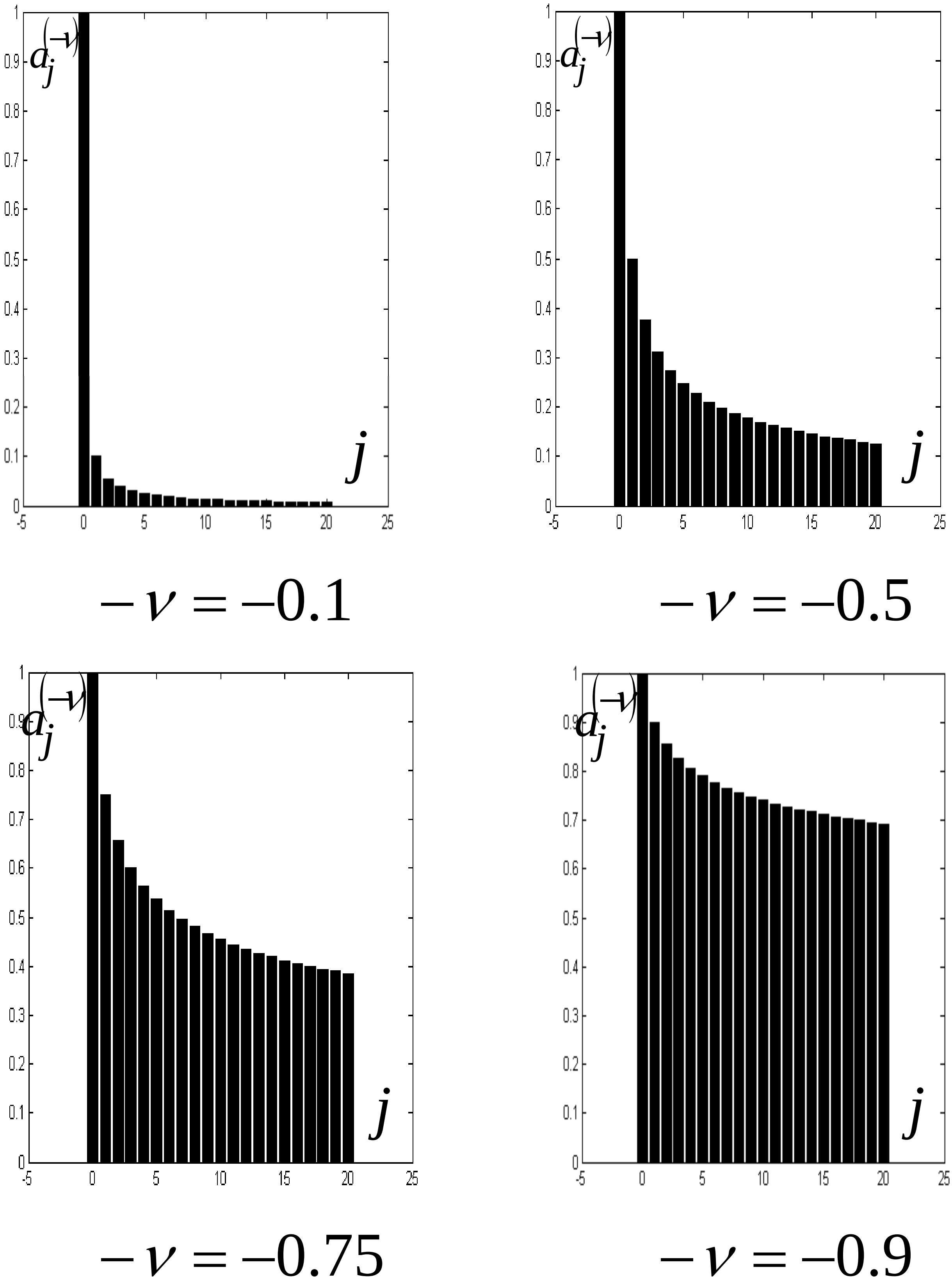


Fig.1.6 Coefficients $a_j^{(-\nu)}$ vs. j
for $\nu = const$ from $(0,1)$.

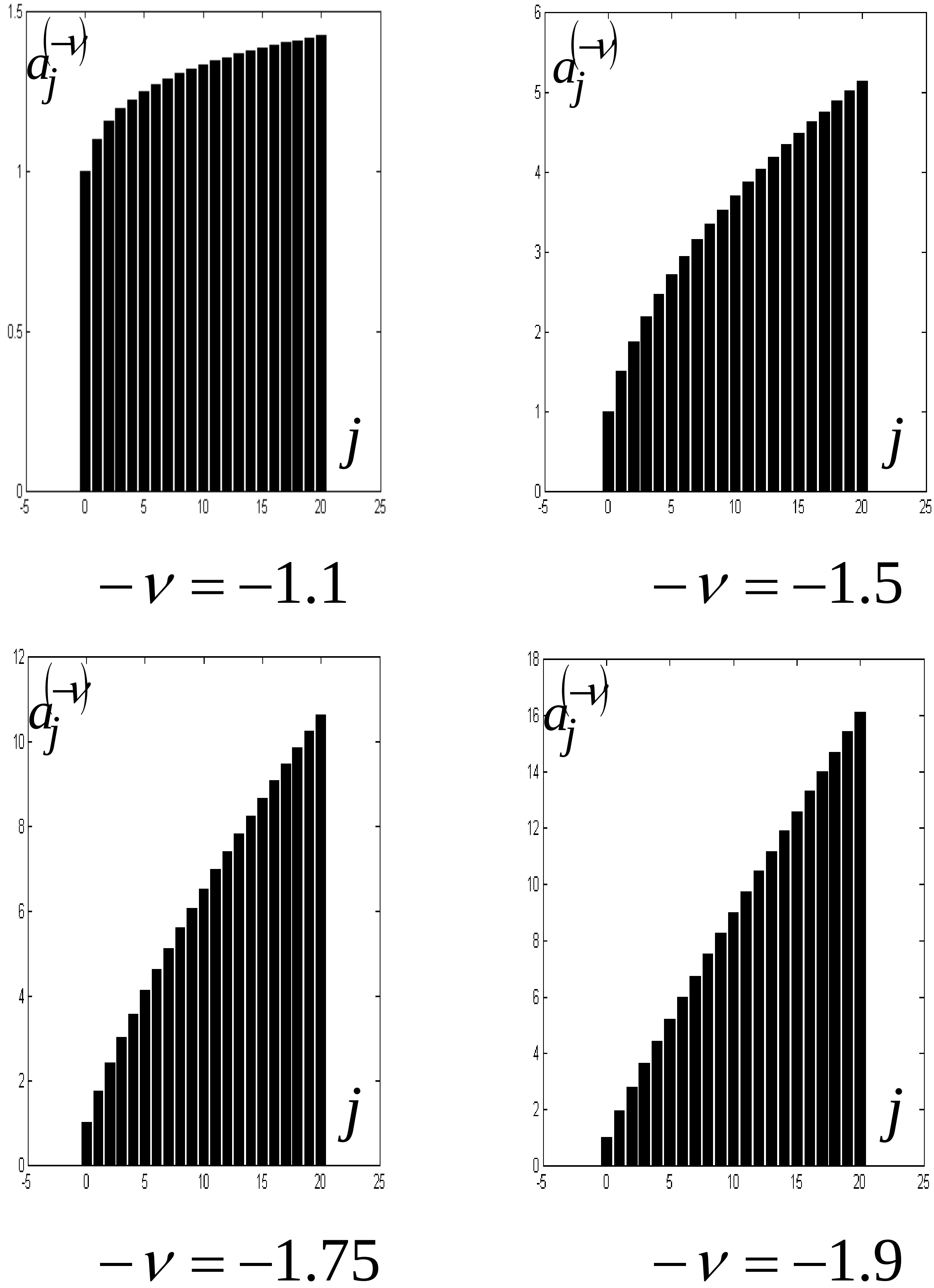


Fig.1.6 Coefficients $a_j^{(-\nu)}$ vs. j
for $\nu = const$ from $(1,2)$.

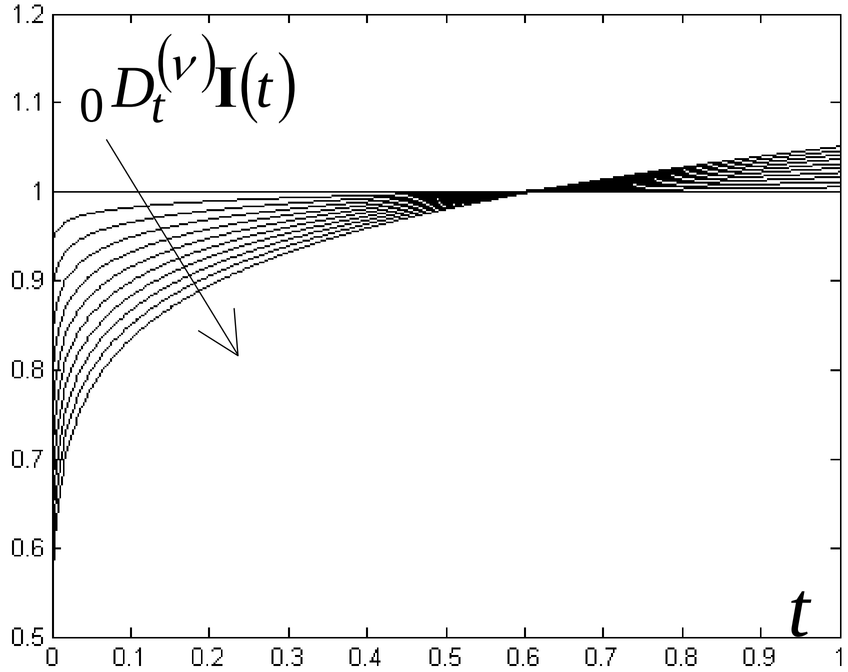


Fig.1.7 FOI of $f(t)=\mathbf{I}(t)$ of orders from a set $\{0.00, \quad -0.01, \quad \cdots \quad -0.08, \quad -0.09\}$.

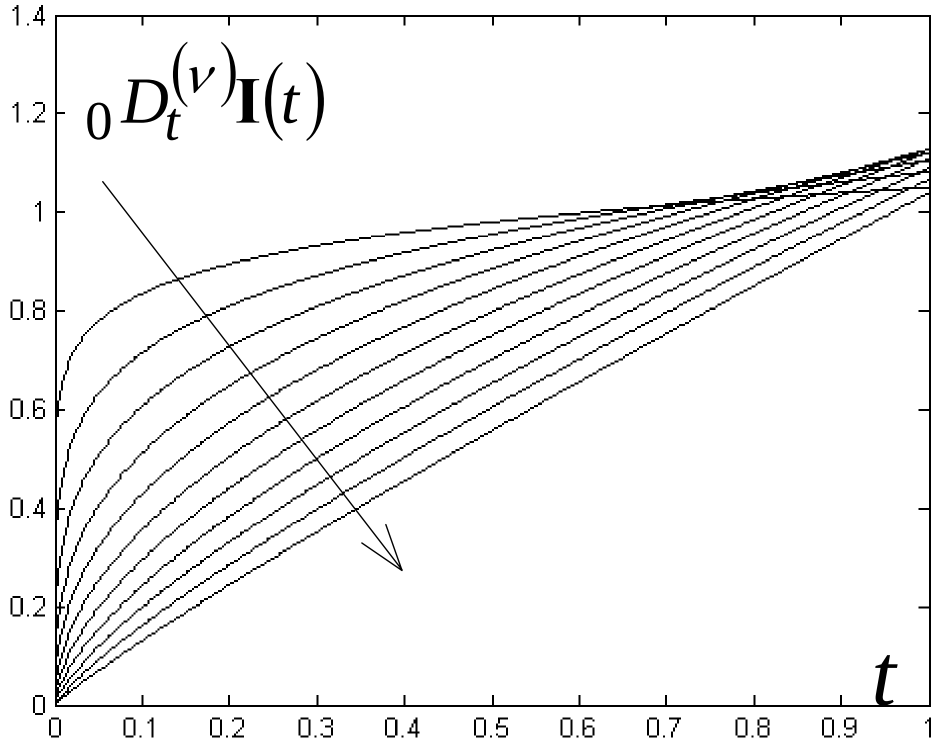


Fig.1.8 FOI of $f(t)=\mathbf{I}(t)$ of orders from a set $\{-0.1, \quad -0.2, \quad \cdots \quad -0.9, \quad -1.0\}$.

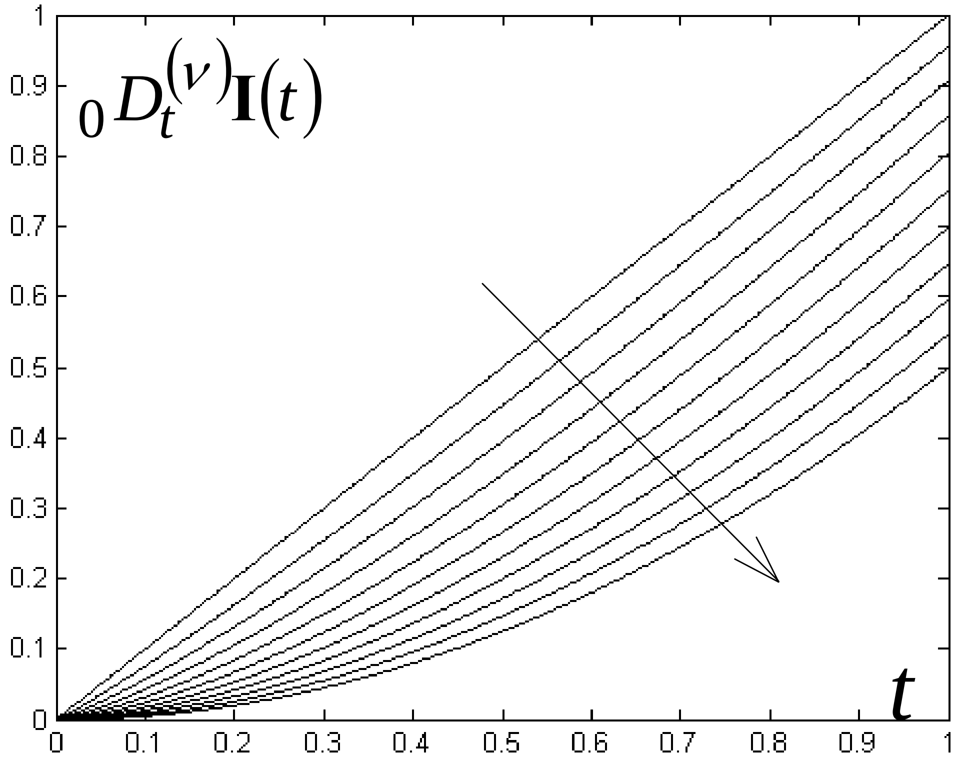


Fig.1.9 FOI of $f(t)=\mathbf{I}(t)$ of orders from a set $\{-1.0,\quad -1.1,\quad \cdots\quad -1.9,\quad -2.0\}$.

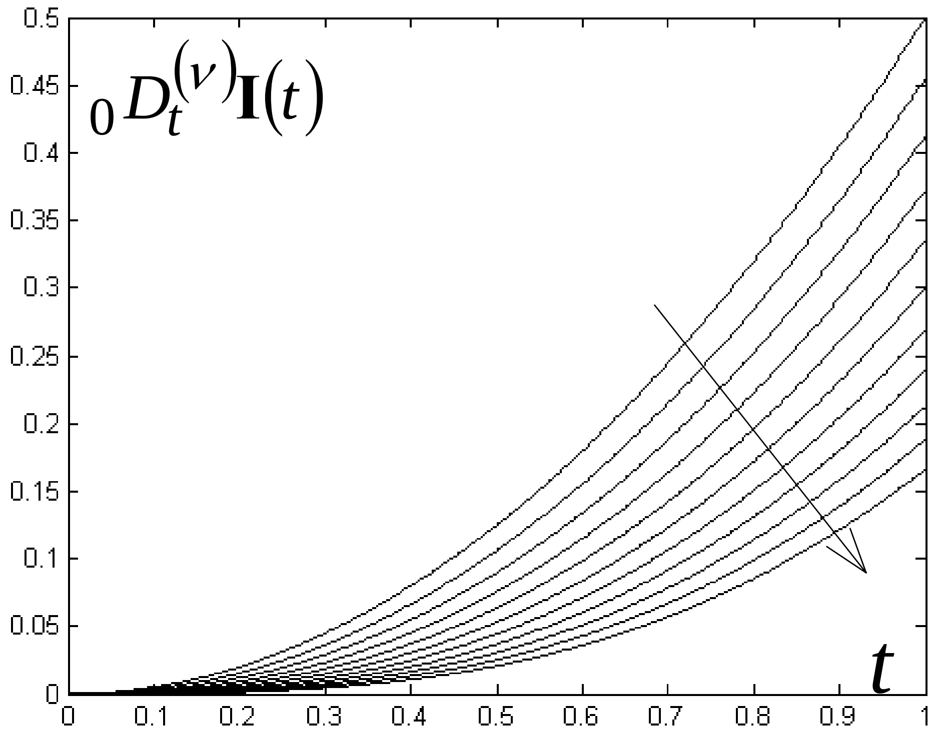


Fig.1.7 FOI of $f(t)=\mathbf{I}(t)$ of orders from a set $\{-2.0,\quad -2.1,\quad \cdots\quad -2.9,\quad -3.0\}$

1.2 Riemann – Liouville FOD/I

For $\nu > 0$ one defines

$$n = [\nu] + 1 \in \mathbf{Z}$$

$$n - 1 \leq \nu < n$$

$$0 \leq n - \nu = \mu < 1$$

Definition 2.1

The Riemann – Liouville left-sided integral of order $\mu > 0$ of a real function $f(t)$ is defined by an integral

$$ {}_{t_0}I_t^{(\mu)} f(t) = \frac{1}{\Gamma(\mu)} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{1-\mu}} d\tau $$

Definition 2.2

The Riemann – Liouville left-sided derivative of order $\nu > 0$ of a real function $f(t)$ is defined by an integral

$$ {}_{t_0}D_t^{(\nu)} f(t) = $$

$$ = \frac{d^n}{dt^n}\left[{}_{t_0}I_t^{(\mu)} f(t)\right] = \frac{d^n}{dt^n}\left[{}_{t_0}I_t^{(n-\nu)} f(t)\right] $$

$$ \frac{1}{\Gamma(n-\nu)} \frac{d^n}{dt^n}\left[\int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{1-n+\nu}} d\tau\right] $$

where:

t_0, t - terminals of an integration satisfying $-\infty < t_0 < t < \infty$,

$n \in \mathbf{Z}$ - integration/differentiation order,

$\Gamma(\nu)$ - Euler gamma function.

$$
{}_{t_0}D_t^{(\nu)} f(t) =
$$

$$
\frac{d^{n-1}}{dt^{n-1}}\left\{\frac{1}{\Gamma(n-\nu)}\frac{d}{dt}\left[\int_{t_0}^{t}(t-\tau)^{n-\nu-1} f(\tau)d\tau\right]\right\}
$$

Next:
integrating by parts

$$
\int_{t_0}^{t}(t-\tau)^{n-\nu-1} f(\tau)d\tau =
$$

$$
\frac{(t-t_0)^{n-\nu}\; f(t_0)}{n-\nu} + \frac{1}{n-\nu}\int_{t_0}^{t}(t-\tau)^{n-\nu} f^{(1)}(\tau)d\tau
$$

and differentiating

$$\frac{d}{dt}\left[\int_{t_0}^{t}(t-\tau)^{n-\nu-1}f(\tau)d\tau\right]=$$

$$(t-t_0)^{n-\nu-1}f(t_0)+\int_{t_0}^{t}(t-\tau)^{n-\nu-1}f^{(1)}(\tau)d\tau$$

one gets

$${}_{t_0}D_t^{(\nu)}f(t)=$$

$$=\frac{d^{n-1}}{dt^{n-1}}\left[\frac{(t-t_0)^{n-\nu-1}f(t_0)}{\Gamma(n-\nu)}+\frac{1}{\Gamma(n-\nu)}\int_{t_0}^{t}(t-\tau)^{n-\nu-1}f^{(1)}(\tau)d\tau\right]$$

Equivalent forms of Riemann – Liouville left – sided fractional – order derivative

$$ {}_{t_0}D_t^{(\nu)} f(t) = $$

$$ = \sum_{i=0}^{n-1} \frac{(t-t_0)^{i-\nu} f^{(i)}(t_0)}{\Gamma(i+1-\nu)} + $$

$$ \frac{1}{\Gamma(n-\nu)} \int_{t_0}^{t} (t-\tau)^{n-\nu-1} f^{(n)}(\tau) d\tau $$

$$ {}_{t_0}D_t^{(\nu)} f(t) = $$

$$ = \frac{1}{\Gamma(n-\nu)} \int_{t_0}^{t} (t-\tau)^{n-\nu-1} f^{(n)}(\tau) d\tau + $$

$$ = \begin{bmatrix} \frac{(t-t_0)^{n-1-\nu}}{\Gamma(n-\nu)} & \frac{(t-t_0)^{n-2-\nu}}{\Gamma(n-1-\nu)} & \cdots & \frac{(t-t_0)^{-\nu}}{\Gamma(1-\nu)} \end{bmatrix} \begin{bmatrix} f^{(n-1)}(t_0) \\ f^{(n-2)}(t_0) \\ \vdots \\ f^{(0)}(t_0) \end{bmatrix} $$

1.3 Caputo fractional - order derivative / integral (RL-FOD/I)

One assumes that:

- ${}_{t_0}D_t^{(\nu)} f(t)$ exists,

- ${}_{t_0}D_t^{(\nu)} (t-t_0)^i = \frac{\Gamma(i+1)}{\Gamma(i+1-\nu)} (t-t_0)^{i-\nu}$

Definition 3.1

The Caputo left-sided derivative of order $\nu > 0$ of a real function $f(t)$ is defined by an integral

$$ {}_{t_0}^{C}D_t^{(\nu)} f(t) = \frac{1}{\Gamma(n-\nu)} \int_{t_0}^{t} (t-\tau)^{n-\nu-1} f^{(n)}(\tau) d\tau $$

t_0, t - terminals of an integration satisfying $-\infty < t_0 < t < \infty$,

$n-1 < \nu < n$, $n \in \mathbb{Z}$ differentiation order,

$\Gamma(\nu)$ - Euler gamma function.

To the above integral one adds and subtracts a following expression

$$\sum_{i=0}^{n-1} \frac{f^{(i)}(t_0)}{\Gamma(i+1-\nu)} (t-t_0)^{i-\nu}$$

one gets

$$ {}_{t_0}^{C}D_t^{(\nu)} f(t) = $$

$$ {}_{t_0}D_t^{(\nu)} f(t) - \sum_{i=0}^{n-1} \frac{f^{(i)}(t_0)}{\Gamma(i+1-\nu)} (t-t_0)^{i-\nu}$$

or

$$ {}_{t_0}^{C}D_t^{(\nu)} f(t) = {}_{t_0}D_t^{(\nu)} \left[f(t) - \sum_{i=0}^{n-1} \frac{f^{(i)}(t_0)}{i!} (t-t_0)^{i} \right]$$

Finally

$$ {}_{t_0}^{C}D_t^{(\nu)} f(t) = {}_{t_0}D_t^{(\nu)} f(t)$$

for $f^{(i)}(t_0) = 0 \quad i = 0,1,2,\cdots,n-1$.