1.4 Fundamental properties of FOD/I

Linearity

Consider a real, bounded functions $f_1(t)$ and $f_2(t)$, having ${}_{t_0}D_t^{(\nu)} f_1(t)$ and ${}_{t_0}D_t^{(\nu)} f_2(t)$. Then for α_1 i α_2

$$
{}_{t_0}D_t^{(\nu)}\left[\alpha_1 f_1(t) + \alpha_2 f_2(t)\right] =
$$

$$
\alpha_1 {}_{t_0}D_t^{(\nu)} f_1(t) + \alpha_2 {}_{t_0}D_t^{(\nu)} f_2(t)
$$

Compositions of FOD/Ss

a) For $\nu_1, \nu_2 \in \mathrm{R}_+$ and function $f(t) \in L_p(t_0, t)$, $1 \leq p \leq \infty$

$$
{}_{t_0}I_t^{(\nu_1)}\left[{}_{t_0}I_t^{(\nu_2)} f(t)\right] = {}_{t_0}I_t^{(\nu_2+\nu_1)} f(t)
$$

exist at almost every point of $[t_0, t]$ and at any point of $[t_0, t]$ if $\nu_1 + \nu_2 > 1$.

b) For $\nu_1,\nu_2 \in \mathrm{R}_+$ and function $f(t) \in L_p(t_0,t)$, $1 \le p \le \infty$

$$ {}_{t_0}D_t^{(\nu_1)}\left[{}_{t_0}I_t^{(\nu_2)} f(t)\right] = {}_{t_0}I_t^{(\nu_2-\nu_1)} f(t) $$

exist at almost every point of $[t_0,t]$.

c) For $\nu_1,\nu_2 \in \mathrm{R}_+$, $n_1 - 1 < \nu_1 \le n_1$, $n_2 - 1 < \nu_2 \le n_2$, $n_1,n_2 \in \mathbf{Z}_+$, $\nu_1 + \nu_2 < n_1$ and function $f(t) \in L_1(t_0,t)$,

$$ {}_{t_0}D_t^{(\nu_1)}\left[{}_{t_0}D_t^{(\nu_2)} f(t)\right] = $$

$$ {}_{t_0}D_t^{(\nu_2+\nu_1)} f(t) - $$

$$ \sum_{j=1}^{n_2} \frac{(t-t_0)^{-i-\nu_1}}{\Gamma(1-j-\nu_1)_{t_0}} \left[D_t^{(\nu_2-j)} f(t)\right]_{t=t_0} $$

$$ {}_{t_0}D_t^{(\nu_1)}\left[{}_{t_0}D_t^{(\nu_2)} f(t)\right] \ne {}_{t_0}D_t^{(\nu_2)}\left[{}_{t_0}D_t^{(\nu_1)} f(t)\right] $$

1.6 One – sided Laplace transform of the FOD/S

The one – sided Laplace transform of the FOI

$$\mathcal{L}\left\{ {}_0 I_t^{(\nu)} f(t) \right\} = \frac{F(s)}{s^{\nu}}$$

The one – sided Laplace transform of the FOD

$$\mathcal{L}\left\{ {}_0 D_t^{(\nu)} f(t) \right\} =$$

$$s^{\nu} F(s) - \sum_{k=0}^{n-1} s^{k} \left[{}_0 D_t^{(\nu-k-1)} f(t) \right]_{t=0}$$

where $n-1 \leq \nu < n$

Initial conditions

$$\left[{}_0 D_t^{(\nu-1)} f(t) \right]_{t=0}, \left[{}_0 D_t^{(\nu-2)} f(t) \right]_{t=0} \cdots$$

$$\cdots \left[{}_0 D_t^{(\nu-n)} f(t) \right]_{t=0}$$

The one – sided Laplace transform of the Caputo FOD

$$\mathcal{L}\left\{ {}_{0}^{C}D_{t}^{(\nu)} f(t) \right\} =$$
$$s^{\nu} F(s) - \sum_{k=0}^{n-1} s^{\nu-1-k} f^{(k)}\left(0^{+}\right)$$

Initial conditions

$$\left[{}_{0}D_{t}^{(0)} f(t) \right]_{t=0} = f(0),$$
$$\left[{}_{0}D_{t}^{(1)} f(t) \right]_{t=0} = f'(0) \quad \ldots$$
$$\ldots \left[{}_{0}D_{t}^{(n)} f(t) \right]_{t=0} = f^{n}(0)$$

$$\mathcal{L}\left\{t^{\alpha}\mathbf{1}(t)\right\}=\frac{\Gamma(\alpha+1)}{s^{\alpha+1}} \text{ for } \alpha>-1,$$

$$\mathcal{L}^{-1}\left\{\frac{\Gamma(\nu+1)}{s^{\nu+1}}\right\}=t^{\nu} \text{ for } \nu>-1,$$

$$\mathcal{L}^{-1}\left\{\frac{k!s^{\nu-\mu}}{\left(s^{\nu}\pm a\right)^{k+1}}\right\}=t^{\nu k+\mu-1}E_{\nu,\mu}^{(k)}\left(\mp at^{\nu}\right)$$

$$\text{for } k\in \mathrm{Z}_{+}, \ \mathrm{Re}\{s\}>|a|^{\nu^{-1}},$$

where

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\alpha k+\beta)}, \ z,\beta\in \mathrm{C}, \ \mathrm{Re}\{\alpha\}>0$$

two – parameter Mittag – Leffler function

$$E_{\nu,\mu}^{(k)}\left(\pm at^{\nu}\right)=\frac{d}{dt^{k}}E_{\nu,\mu}\left(\pm at^{\nu}\right).$$

For $k=0,\ \mu=1,\ \nu>0$

$$\mathcal{L}^{-1}\left\{\frac{s^{\nu-1}}{s^{\nu}\pm a}\right\}=E_{\nu,1}\left(\mp at^{\nu}\right)$$

For $k=0,\ \mu=1,\ \nu=1$

$$\mathcal{L}^{-1}\left\{\frac{1}{s\pm a}\right\}=E_{1,1}(\mp at)=e^{\mp at}$$

For $k=0,\ \nu=1$ and $\mu=\kappa+1$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^{\kappa}(s\pm a)}\right\}=t^{\kappa}E_{1,\kappa+1}(\mp at)$$