FRACTIONAL CALCULUS IN AUTOMATICS

Piotr Ostalczyk
Institute of Applied Computer Science
Lodz University of Technology
Poland

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Contens

2. Fractional discrete – time system

Fractional – order backward difference/ sum Grünwald – Letnikov FOBD/S

Definition 3.1

Given a discrete - variable bounded real function $f(k)$. The Grünwald – Letnikov fractional – order backward difference (GL-FOBD) of order $\nu \in \mathbf{R}_+ / \mathbf{Z}_+$ is defined as a finite sum

$$ {}_{k_0}\Delta_k^{(\nu)} f(k) = \sum_{i=k_0}^{k} a_{i-k_0}^{(\nu)} f_{k-i+k_0} $$

with coefficients $a_i^{(\nu)}$

$$ a_i^{(\nu)} = \begin{cases} 0 & \text{for} \quad i < 0 \\ 1 & \text{for} \quad i = 0 \\ (-1)^i \dfrac{\nu(\nu-1)\cdots(\nu-i+1)}{i!} & \text{for} \quad i = 1,2,\cdots \end{cases} $$

$${}_{k_0}\Delta_k^{(\nu)} f(k) =$$

$$\begin{bmatrix} a_0^{(\nu)} & a_1^{(\nu)} & \cdots & a_{k-k_0}^{(\nu)} \end{bmatrix} \begin{bmatrix} f_k \\ f_{k-1} \\ \vdots \\ f_{k_0} \end{bmatrix} = \begin{bmatrix} {}_0\mathbf{a}_{k-k_0}^{(\nu)} \end{bmatrix}^{\mathrm{T}} {}_k\mathbf{f}_{k_0}$$

Definition 3.2 (GL-FOBS)

Given a discrete - variable bounded real function $f(k)$, The Grünwald – Letnikov fractional – order backward sum (GL-FOBS) of order $\nu \in \mathbf{R}_+ / \mathbf{Z}_+$ is defined as a finite sum

$${}_{k_0}\Delta_k^{(-\nu)} f(k) = \sum_{i=k_0}^{k} a_{i-k_0}^{(-\nu)} f_{k-i+k_0}$$

$${}_{k_0}\Delta_k^{(\nu)} f(k) =$$

$$\begin{bmatrix} a_0^{(\nu)} & a_1^{(\nu)} & \cdots & a_{k-k_0}^{(\nu)} \end{bmatrix} \begin{bmatrix} f_k \\ f_{k-1} \\ \vdots \\ f_{k_0} \end{bmatrix} = \begin{bmatrix} {}_0\mathbf{a}_{k-k_0}^{(\nu)} \end{bmatrix}^{\mathrm{T}} {}_k\mathbf{f}_{k_0}$$

Riemann – Liouville FOBD/S

One assumes

$$n \le \nu < n+1, \text{ where } n \in \mathbf{Z}_+.$$

Definition 3.3

Given a discrete - variable bounded real function $f(k)$. The Riemann - Liouville fractional – order backward difference (RL-FOBD) of order $\nu \in \mathbf{R}_+ / \mathbf{Z}_+$ is defined as a finite sum

$$
{}_{k_0}\Delta_k^{(\nu)} f(k) =
$$

$$
\left[{}_0\mathbf{a}_{k-k_0}^{(\nu-n-1)}\right]^{\mathrm{T}}
\begin{bmatrix}
\Delta^{(n+1)} f(k) \\
\Delta^{(n+1)} f(k-1) \\
\Delta^{(n+1)} f(k-2) \\
\vdots \\
\Delta^{(n+1)} f(k_0)
\end{bmatrix} +
$$

$$
\begin{bmatrix} a_k^{(\nu-1)} & a_k^{(\nu-2)} & a_k^{(\nu-3)} & \cdots & a_k^{(\nu-n-1)} \end{bmatrix}
\begin{bmatrix}
f(k_0-1) \\
\Delta^{(1)} f(k_0-1) \\
\Delta^{(2)} f(k_0-1) \\
\vdots \\
\Delta^{(n)} f(k_0-1)
\end{bmatrix},
$$

$$
{}_{k_0}\Delta_k^{(\nu)} f(k) =
$$

$$
\sum_{i=k_0}^{k} a_{i-k_0}^{(\nu-n-1)} \Delta^{(n+1)} f(k + k_0 - i) +
$$

$$
\sum_{i=0}^{n} a_k^{(\nu-1-i)} \Delta^{(i)} f(k_0 - 1)
$$

Horner form of the FOBD/S

Definition 3.4
Given a discrete - variable bounded real function $f(k)$. The Horner fractional – order backward difference (H-FOBD) of order $\nu \in \mathbf{R}_+ / \mathbf{Z}_+$ is defined as a finite sum

$$ {}_0\Delta_k^{(\nu)} y_k = $$

$$ = c_0^{(\nu)}\Big[y_k + c_1^{(\nu)}\Big[y_{k-1} + c_2^{(\nu)}[y_{k-2} + \cdots $$

$$ + \cdots c_{k-2}^{(\nu)}\Big[y_2 + c_{k-1}^{(\nu)}\Big[y_1 + c_k^{(\nu)} y_0\Big]\Big]\cdots\Big]\Big] $$

where

$$ c_i^{(\nu)} = \begin{cases} 1 & \text{for} \quad i = 0 \\ \dfrac{i-1-\nu}{i} & \text{for} \quad i = 1,2,3,\cdots \end{cases} $$

The one – sided $\mathcal{Z}$ transform of the FOBD

$$\mathcal{Z}\left\{{}_0\Delta_k^{(\nu)} f_k\right\} = \left(1\text{-}z^{-1}\right)^{\nu} F(z) +$$

$$\begin{bmatrix} a_1^{(\nu)} & a_2^{(\nu)} & a_3^{(\nu)} & \cdots \end{bmatrix} \begin{bmatrix} z^0 & 0 & 0 & \cdots \\ z^{-1} & z^0 & 0 & \cdots \\ z^{-2} & z^{-1} & z^0 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} f_{-1} \\ f_{-2} \\ f_{-3} \\ \vdots \end{bmatrix}$$

The one – sided $\mathcal{Z}$ transform of the FOBS

$$\mathcal{Z}\left\{{}_0\Delta_k^{(-\nu)} f_k\right\} = \left(1\text{-}z^{-1}\right)^{-\nu} F(z)$$

3.1 Discrete transfer function of a fractional system

Consider a linear, time – invariant difference equation

$$\sum_{i=0}^{n} A_i \; {}_0\Delta k_t^{(\nu_i)} y_k = \sum_{j=0}^{m} B_j \; {}_0\Delta_k^{(\mu_j)} u_k$$

$A_i = const \in \mathsf{R}$, $i = 1, 2, \cdots n-1$,
$B_j = const \in \mathsf{R}$, $j = 1, 2, \cdots m$,
$m \le n$, $m, n \in \mathsf{Z}_+$,

$A_n = 1$,

$\nu_n > \nu_{n-1} > \cdots > \nu_1 > \nu_0 = 0,$

$\mu_n > \mu_{n-1} > \cdots > \mu_1 > \mu_0 = 0,\ \nu, \mu \in \mathsf{R}_+,$

${}_0\Delta_k^{(\nu_i)} y_k,\ {}_0\Delta_k^{(\nu_i)} u_k$ - FOBDs of input and output functions,

y_{-i} for $i = 1, 2, 3, \cdots,$ initial conditions,

$$\mathcal{Z}\left\{\sum_{i=0}^{p} A_i \; {}_0\Delta_k^{(\nu_i)} y_k\right\} = \mathcal{Z}\left\{\sum_{j=0}^{q} B_j \; {}_0\Delta_k^{(\mu_j)} u_k\right\}$$

with zero initial conditions, $\mathcal{Z}\{y(k)\} = Y(z)$ $\mathcal{Z}\{u(k)\} = U(z)$

$$G(z) = \frac{Y(z)}{U(z)} = \frac{\sum_{j=0}^{q} B_j \left(1 - z^{-1}\right)^{(\mu_j)}}{\sum_{j=0}^{p} A_j \left(1 - z^{-1}\right)^{(\nu_j)}}$$

$0 < \mu_1 < \mu_2 < \cdots < \mu_q,$

$0 < \nu_1 < \nu_2 < \cdots < \nu_p,$

$\mu_q < \nu_p$

$A_i, B_i \in \mathrm{R}$

Characterisitc equation

$$w(z) = z^{\nu_p} \sum_{j=0}^{p} A_j \left[z^{-1}(z-1)\right]^{\nu_j} =$$

$$z^{\nu_p} \sum_{j=0}^{p} A_j \left[z^{-\nu_j} (z-1)^{\nu_j} \right] =$$

$$\sum_{j=0}^{p} A_j \left[z^{\nu_p - \nu_j} (z-1)^{\nu_j} \right]$$

Example 3.1

$$G(z) = \frac{Y(z)}{U(z)} = \frac{3\left(1-z^{-1}\right)^{0.5} + 4}{\left(1-z^{-1}\right)^{1.5} + 2\left(1-z^{-1}\right) + 1}$$

$$\left(1-z^{-1}\right)^{\nu} = \sum_{i=0}^{\infty} a_i^{(\nu)} z^{-i}$$

$$G(z) = \frac{Y(z)}{U(z)} = \frac{3 \sum_{i=0}^{\infty} a_i^{(0.5)} z^{-i} + 4}{\sum_{i=0}^{\infty} a_i^{(1.5)} z^{-i} + 2\left(1-z^{-1}\right) + 1} =$$

$$G(z) = \frac{Y(z)}{U(z)} = \frac{\sum_{i=0}^{\infty} d_i \, z^{-i}}{\sum_{i=0}^{\infty} e_i \, z^{-i}}$$

Assuming that

$$a_i^{(\nu)} = 0 \text{ for } i = L+1, L+2, \cdots$$

$$\left(1 - z^{-1}\right)^{\nu} = \sum_{i=0}^{\infty} a_i^{(\nu)} z^{-i} \approx \sum_{i=0}^{L} a_i^{(\nu)} z^{-i}$$

one gets

$$G(z) = \frac{Y(z)}{U(z)} = \frac{\sum_{i=0}^{L} d_i \, z^{-i}}{\sum_{i=0}^{L} e_i \, z^{-i}}$$

(an approximation of the FO discrete system by the classical realization)

3.1.1 Response of a fractional system described by a discrete transfer function

One assumes that $\nu_1, \nu_2, \cdots, \nu_p$, $\mu_1, \mu_2, \cdots, \mu_q$ are national numbers

$$\nu_i = \frac{e_i}{d_i} \text{ for } i = 1,2,\cdots,p,$$

$$\mu_i = \frac{g_i}{f_i} \text{ for } i = 1,2,\cdots,q,$$

n - least common denominator

$$\nu_i = \frac{n_i}{n} \text{ for } i = 1,2,\cdots,p,$$

$$\mu_i = \frac{m_i}{n} \text{ for } i = 1,2,\cdots,q.$$

For $n_i, m_i, n \in \mathrm{Z}_+ \cup \{0\}$

$$G(s)=$$

$$\frac{b_q\left(\left(1-z^{-1}\right)^{\frac{1}{n}}\right)^{m_q}+b_{q-1}\left(\left(1-z^{-1}\right)^{\frac{1}{n}}\right)^{m_{q-1}}+\cdots+b_0}{\left(\left(1-z^{-1}\right)^{\frac{1}{n}}\right)^{n_p}+a_{p-1}\left(\left(1-z^{-1}\right)^{\frac{1}{n}}\right)^{n_{p-1}}+\cdots+a_0}$$

Introduction a new variable

$$w=\left(1-z^{-1}\right)^{\frac{1}{n}}$$

yields

$$F(w)=\frac{b_q w^{m_q}+b_{q-1}w^{m_{q-1}}+\cdots+b_1 w^{m_1}+b_0}{w^{n_p}+a_{p-1}w^{n_{p-1}}+\cdots+a_1 w^{n_1}+a_0}$$

$$=\sum_{i=1}^{n_{p,R}}\frac{R_i}{w+w_{i,R}}+$$

$$\sum_{i=1}^{n_{p,C}}\left(\frac{C_i}{w+w_{i,C}}+\frac{C_i^*}{w+w_{i,C}^*}\right)$$

$n_{p,R}$ - number of real poles,
$w_{i,R}$ - real pole,
$n_{p,C}$ - number of complex poles,
$w_{i,C}$ - complex pole,
R_i - real coefficients,
C_i - complex coefficient.

Example 3.2

Find a unit step response of a system

$$G(z) = \frac{a_0}{\left(1 - z^{-1}\right)^{\frac{1}{n}} + a_0} \cdot \frac{1}{1 - z^{-1}}$$

for $a_0 = \{0.5, \cdots, 1.5\}$, $n = 2$. Substitution

$$w = \left(1 - z^{-1}\right)^{0.5}$$

$$G(w) = \frac{a_0}{w + a_0} \cdot \frac{1}{w^2} = \frac{1}{w^2} - \frac{\frac{1}{a_0}}{w} + \frac{\frac{1}{a_0}}{w + a_0}$$

$$G(z) =$$

$$\frac{1}{1-z^{-1}} - \frac{\frac{1}{a_0}}{\left(1-z^{-1}\right)^{0.5}} + \frac{\frac{1}{a_0}}{\left(1-z^{-1}\right)^{0.5} + a_0}$$

$$f_{1,k} = \mathcal{Z}^{-1}\left\{\frac{z^{-1}}{1-z^{-1}}\right\} = \mathbf{1}_{k-1}$$

$$f_{2,k} = -\frac{1}{a_0}\mathcal{Z}^{-1}\left\{\frac{z^{-1}}{\left(1-z^{-1}\right)^{0.5}}\right\} = -\frac{1}{a_0}a_{k-1}^{(-0.5)}$$

$$f_{3,k} = \frac{1}{a_0}\mathcal{Z}^{-1}\left\{\frac{1}{\left(1-z^{-1}\right)^{0.5} + a_0}\right\} =$$

$$\frac{1}{a_0+1}\left(\delta_{k-1} - \sum_{j=1}^{k-1} a_j^{(\nu)} y_{k-1-j}\right)$$

$$y_k = \sum_{j=1}^{3} f_{j,k}$$

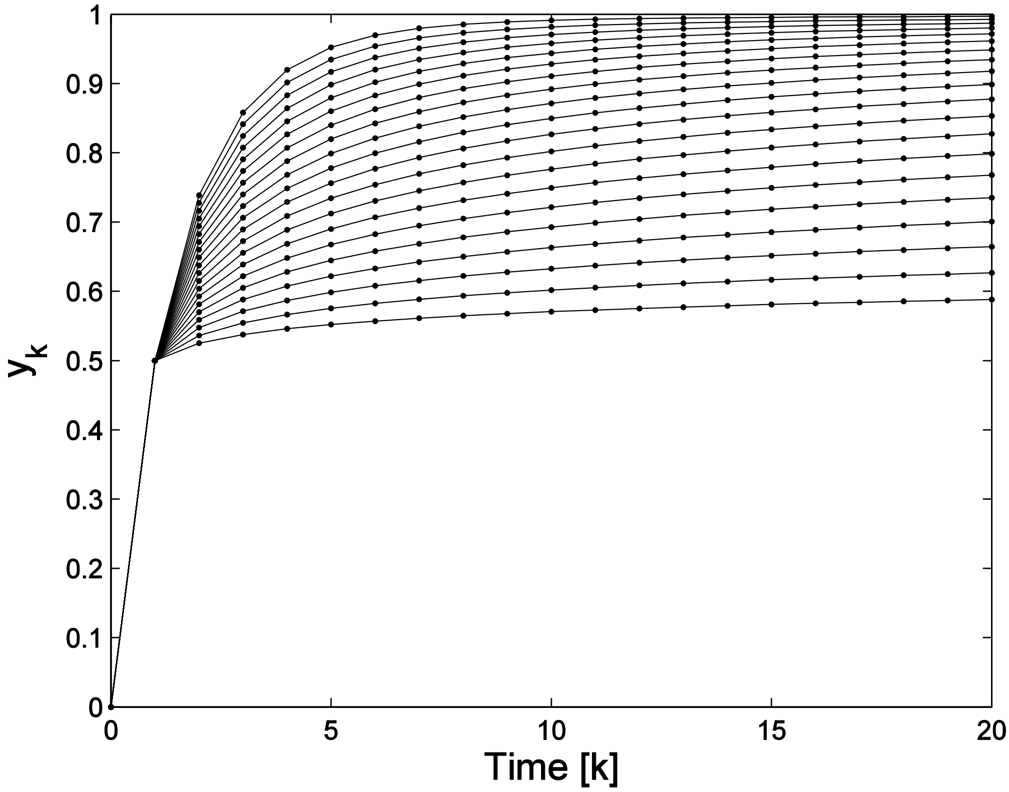

Fig.3.1 Unit step responses for
$a_0 = \{0.1, \cdots, 1.0\}$

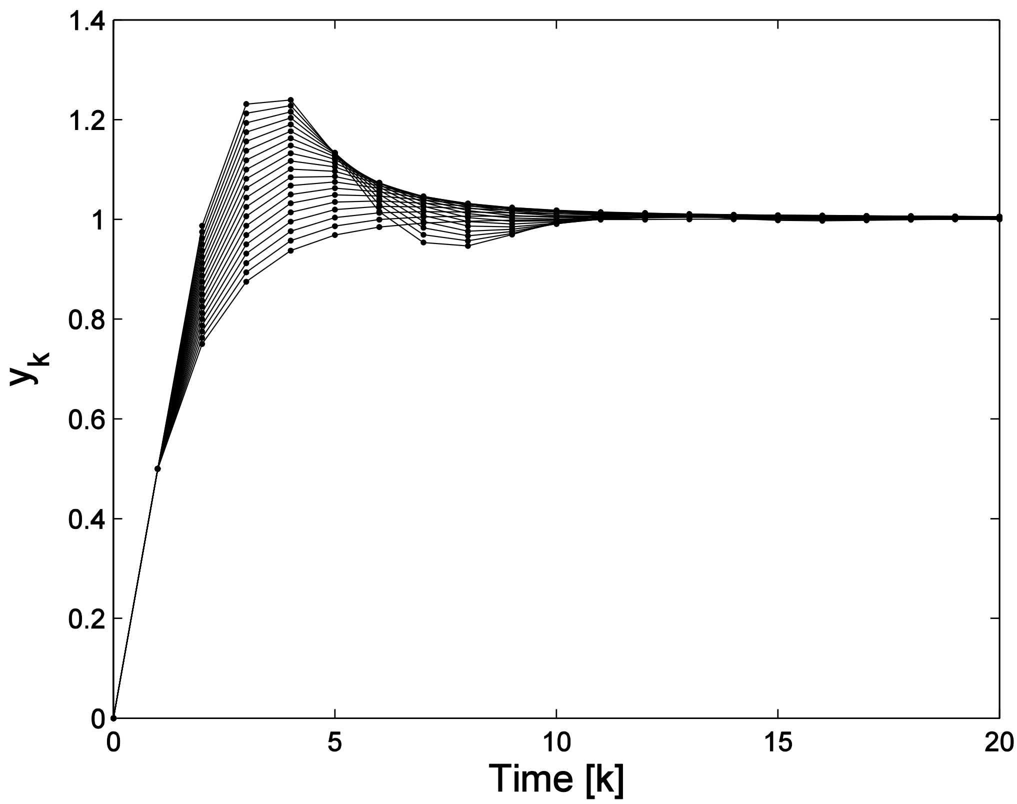

Fig.3.2 Unit step responses for
$a_0 = \{1.1, \cdots, 1.9\}$

3.2 State-space description of a fractional discrete system

$$ {}_0\Delta_k^{(\nu)} \mathbf{x}_k = \mathbf{f}[\mathbf{x}_k, u_k] $$

$$ \boldsymbol{\nu} = \begin{bmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_n \end{bmatrix} \text{ where } 0 < \nu_1 \le \nu_2 \le \cdots \le \nu_n \le 1 $$

$n \times 1$- vector of fractional orders,

$0 < \nu_1 < \nu_2 < \cdots < \nu_n \le 1$
non-commensurate orders
$0 < \nu_1 = \nu_2 = \cdots = \nu_n \le 1$
commensurate orders

$$\mathbf{x}_k = \begin{bmatrix} x_{1,k} \\ x_{2,k} \\ \vdots \\ x_{n,k} \end{bmatrix}$$

$n \times 1$ – state vector

$$_0\Delta_k^{(\nu)}\mathbf{x}(t) = {_0\Delta_k^{(\nu)}} \begin{bmatrix} x_{1,k} \\ x_{2,k} \\ \vdots \\ x_{n,k} \end{bmatrix} = \begin{bmatrix} _0\Delta_k^{(\nu_1)} x_{1,k} \\ _0\Delta_k^{(\nu_2)} x_{2,k} \\ \vdots \\ _0\Delta_t^{(\nu_n)} x_{n,k} \end{bmatrix}$$

$$\mathbf{f}_k = \begin{bmatrix} f_1[\mathbf{x}_k, u_k] \\ f_2[\mathbf{x}_k, u_k] \\ \vdots \\ f_n[\mathbf{x}_k, u_k] \end{bmatrix} = \begin{bmatrix} f_1[x_{1,k}, \cdots x_{n,k}, u_k] \\ f_2[x_{1,k}, \cdots x_{n,k}, u_k] \\ \vdots \\ f_n[x_{1,k}, \cdots x_{n,k}, u_k] \end{bmatrix}$$

SISO linear, time – invariant discrete – time commensurate FO system

$$
{}_0\Delta_{k+1}^{(\nu)}\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{b}u_k
$$

$$
y_k = \mathbf{c}\mathbf{x}_k + du_k
$$

$$
\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n-1} & a_{1n} \\ \vdots & & \vdots & \vdots \\ a_{n-11} & \cdots & a_{n-1n-1} & a_{n-1n} \\ a_{n1} & \cdots & a_{nn-1} & a_{nn} \end{bmatrix}
$$

$$
\mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}, \mathbf{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix}^{\mathrm{T}}, d = [d_1]
$$

$$
0 < v_1 = v_2 = \cdots = v_n = v < 1
$$

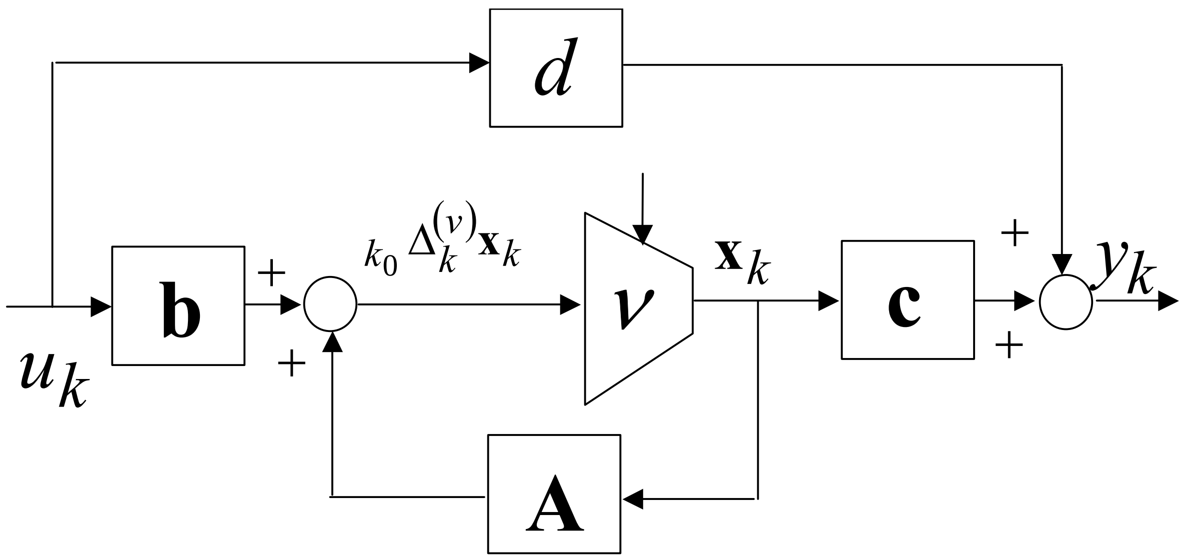

Fig.3.3 Block diagram of the FO linear system with the FODI

$$ {}_0\Delta_{k+1}^{(\nu)}\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{b}u_k $$
$$ y_k = \mathbf{c}\mathbf{x}_k + du_k $$

A substitution of the GL-FOBD definition formula into the state – space equation transforms it to the equivalent form

$$ \mathbf{x}_{k+1} = \mathbf{A}_\nu \mathbf{x}_k + \mathbf{b}u_k - \sum_{i=2}^{k+1} a_i^{(\nu)} \mathbf{x}_{k+1-i} $$

where

$$ \mathbf{A}_\nu = \mathbf{A} + \nu \mathbf{1} $$

2.1.1 Response of a fractional discrete system described by a state space equations

For known $\mathbf{x}_0$, u_k and FO system described by the state – space equation a solution is as follows

$$\mathbf{x}_k = \Phi_k \mathbf{x}_0 + \sum_{i=1}^{k-1} \Phi_{k-1-i} \mathbf{b} u_i$$

where

$$\Phi_k = \begin{cases} \mathbf{1} & \text{for} \quad k = 0 \\ \Phi_{k-1}\mathbf{A}_\nu - \sum_{i=2}^{k} a_i^{(\nu)} \Phi_{k-i} & \text{for} \quad k \geq 1 \end{cases}$$

$$\mathbf{A}_\nu = \mathbf{A} + \nu \mathbf{I}$$

Example 3.4

Evaluate a homogenous response of the FO $\nu = 0.5$ system. System is described by the state – space equations with following matrices and initial conditions

$$\mathbf{A} = \begin{bmatrix} 0.82 & 0.36 \\ -2.44 & -0.62 \end{bmatrix}$$

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Eigenvalues of a matrix **A**

$$z_1 = 0.1 + j0.6$$

$$z_2 = 0.1 - j0.6$$

$$\mathrm{Re}\{z_2\} = 0.1 > 0$$

Asymptotically stable system.

Homogenos system response

$$\mathbf{x}_k = \Phi_k \mathbf{x}_0 =$$

$$\begin{bmatrix} \Phi_{k-1} & \Phi_{k-2} & \cdots & \Phi_0 \end{bmatrix} \begin{bmatrix} \mathbf{A} + \nu \mathbf{I} \\ -a_2^{(\nu)} \mathbf{I} \\ \vdots \\ -a_k^{(\nu)} \mathbf{I} \end{bmatrix} \mathbf{x}_0$$

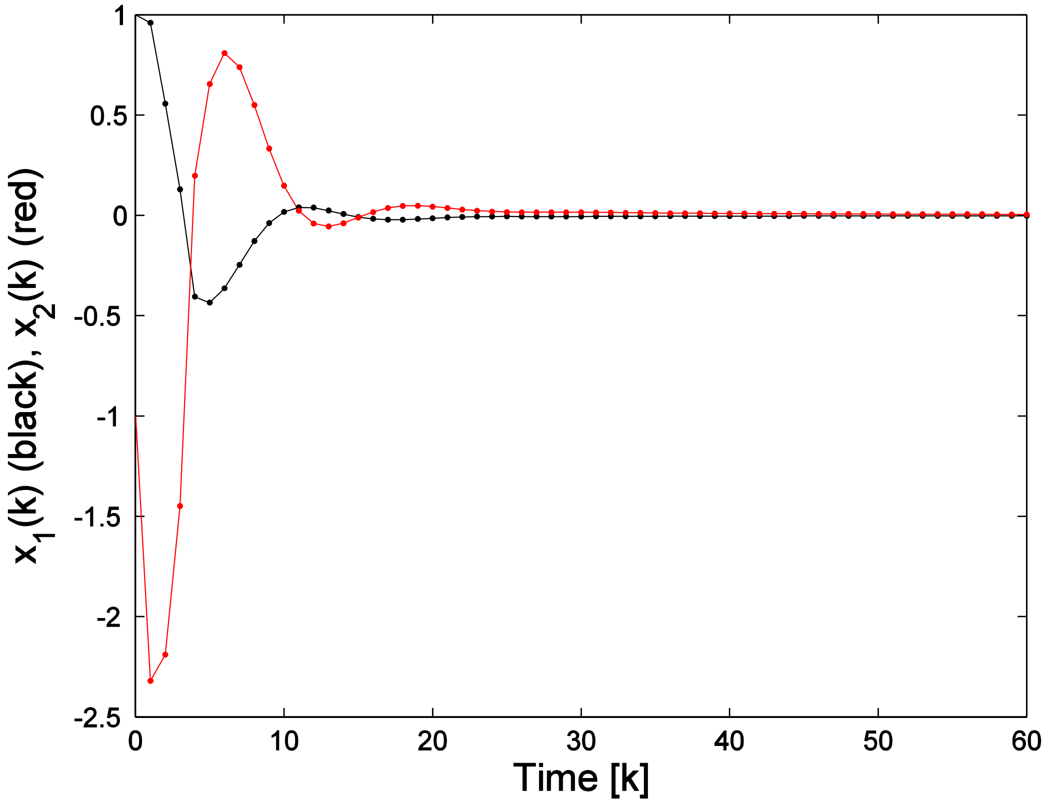

Fig.3.4 FO $\nu = 0.5$ system states vs. discrete time.

For the same state matrix **A** and different order $\nu = 0.68994$

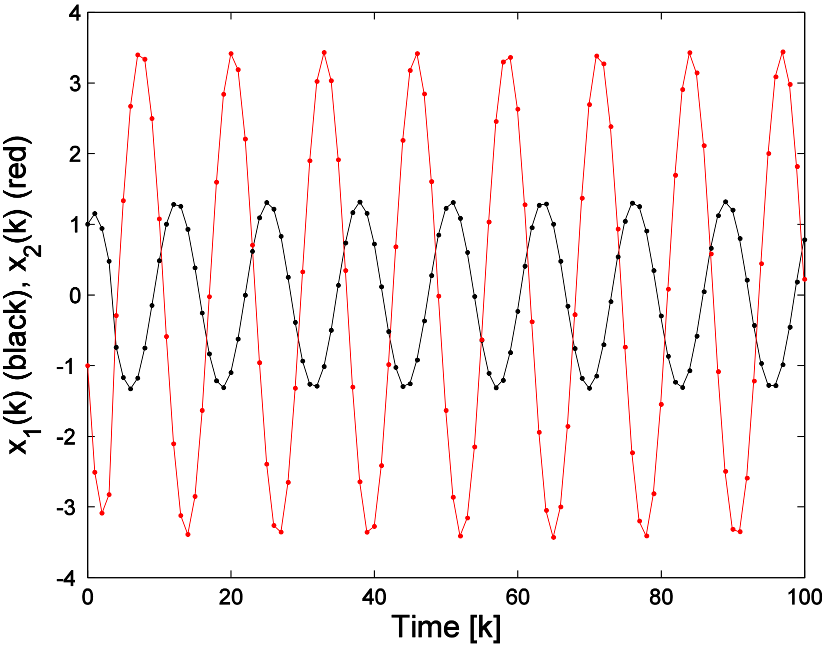

Fig.3.5 FO $\nu = 0.68994$ system states vs. discrete time.

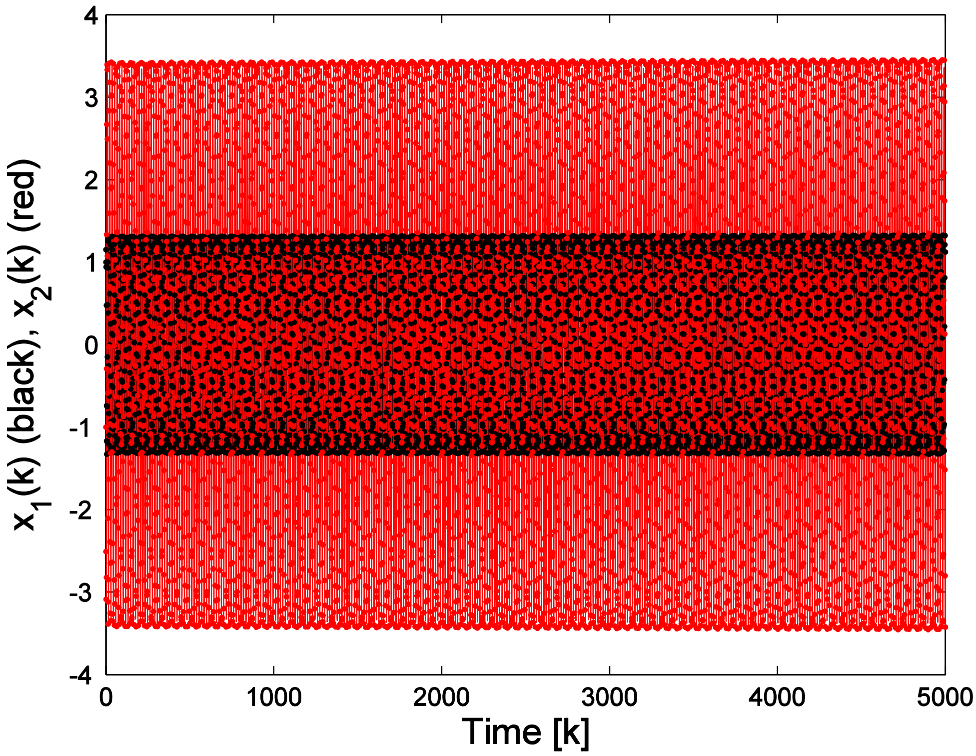

Fig.3.6 FO $\nu = 0.68994$ system states vs. discrete time.

For the same state matrix **A** an order $\nu = 0.75$

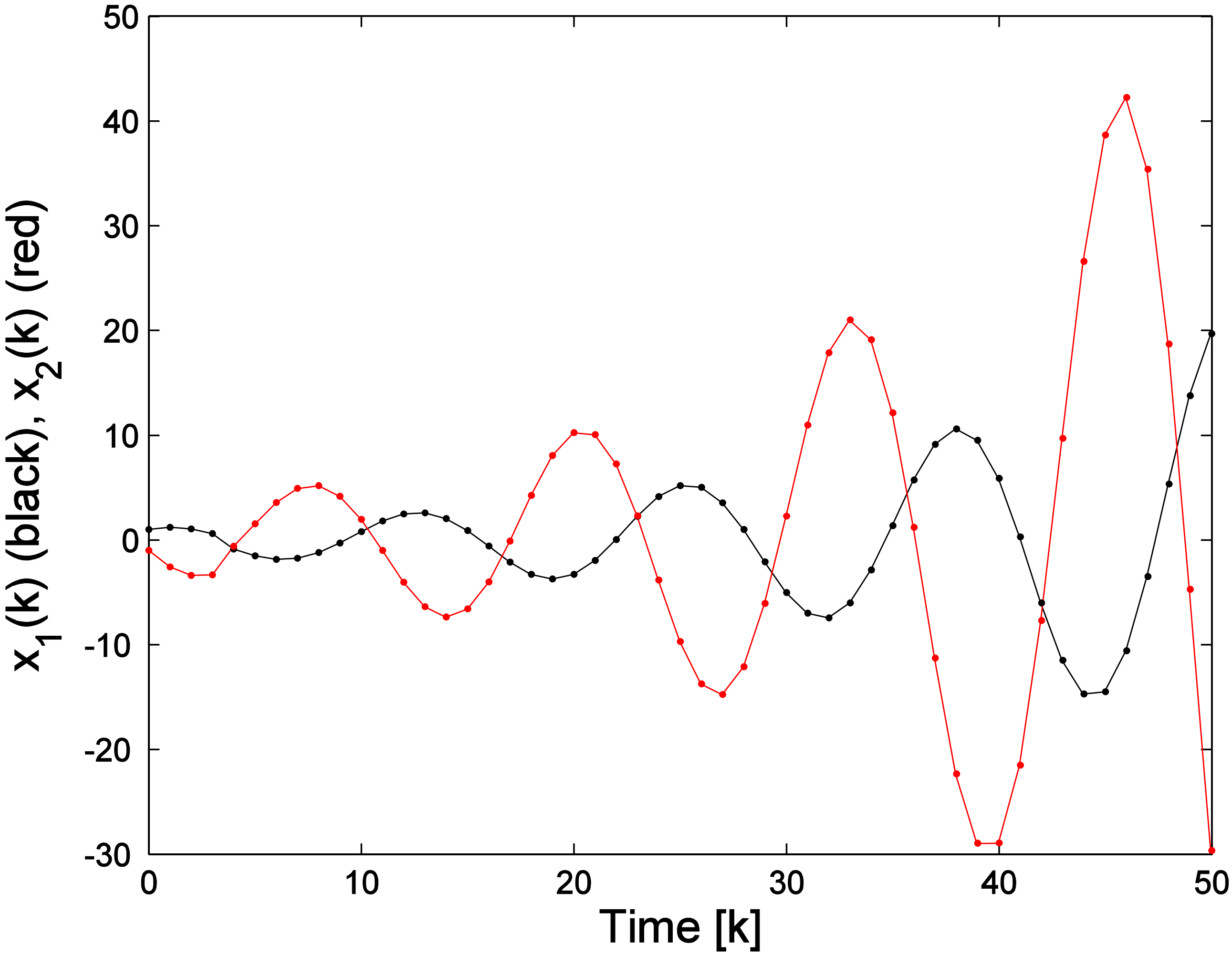

Fig.3.7 FO $\nu = 0.75$ system states vs. discrete time.

3.3 Frequency characteristics of fractional discrete elements

$$G(j\omega) = G(z)\big|_{z=e^{j\omega}}$$

$$\left(1 - z^{-1}\right)^{\nu}\Big|_{z=e^{j\omega}} = \left(1 - e^{-j\omega}\right)^{\nu}$$

3.3 Frequency characteristics of the fractional discrete integrator (FDI)

FDI transfer function

$$G_I(z) = \frac{Y(z)}{U(z)} = \frac{1}{T_u(1 - z^{-1})^{\nu}}$$

Nyquist charasteristics of the FI

$$G_I(j\omega) = \frac{Y(j\omega)}{U(j\omega)} = \frac{1}{T_u\left(1 - e^{-j\omega}\right)^{\nu}}$$

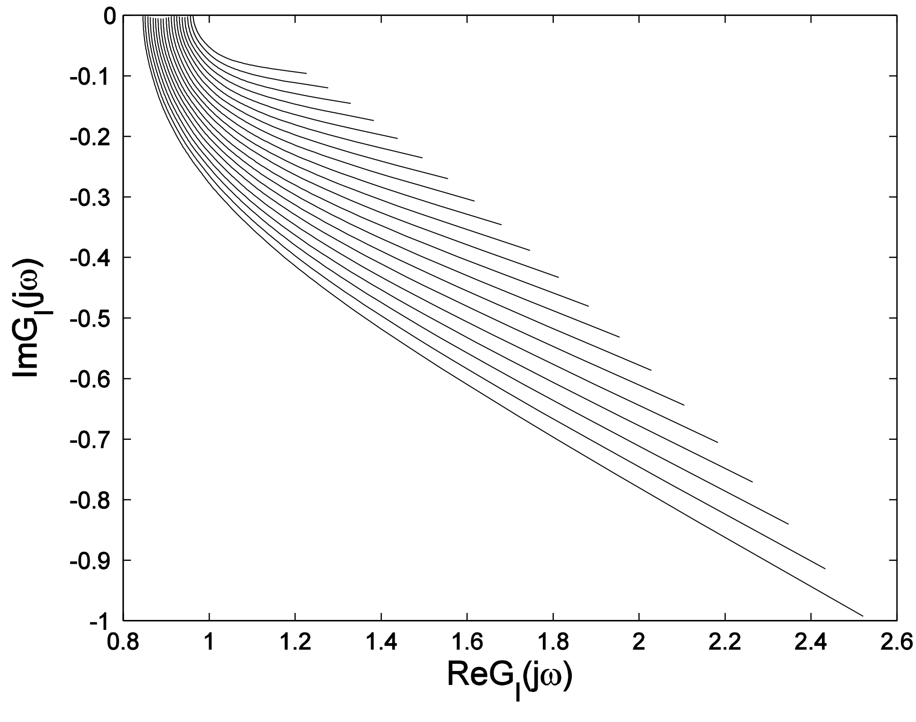

Fig.3.8 Nyquist charastristics of the FDI for different orders $\nu \in [0.05, \cdots 0.25]$ and constant T_u

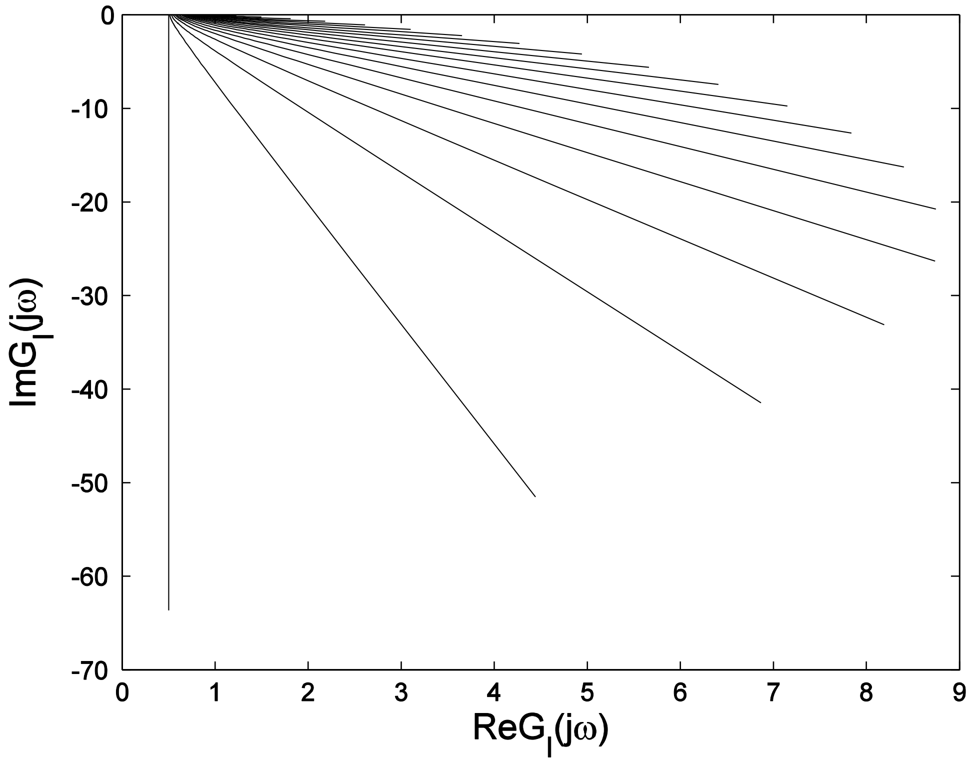

Fig.3.9 Nyquist charastristics of the FI for different orders $\nu \in [0.05, \cdots 0.95, 1.0]$ and constant T_u

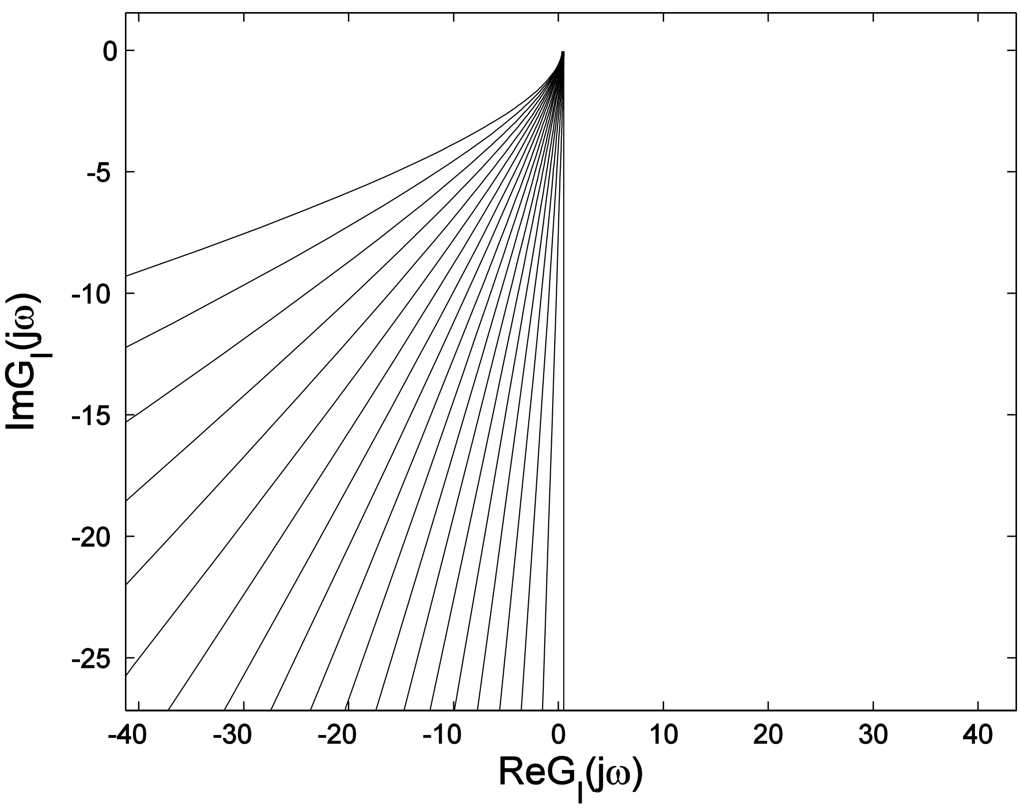

Fig.3.10 Nyquist charastristics of the FDI for different orders $\nu \in [1.0, \cdots 1.95, 2.0]$ and constant T_u

The amplitude characteristic of the FDI

$$\left|G_I(j\omega)\right| = \frac{1}{T_u\left(2\sin\frac{\omega}{2}\right)^{\nu}}$$

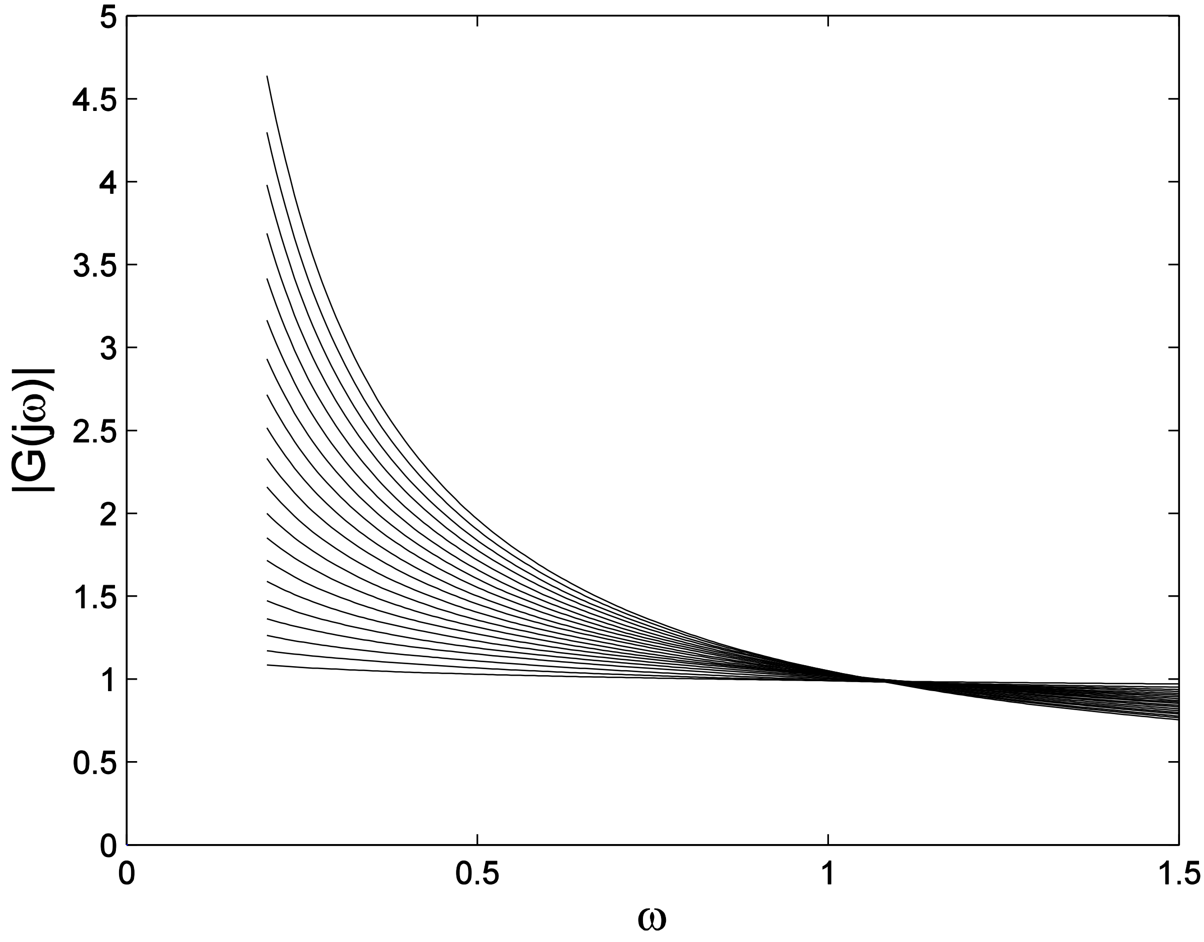

Fig.3.11 Amplitude characteristic of the FI for different orders $\nu \in \left[0.05, \cdots 0.95, 1.0\right]$ and constant $T_u = 1$

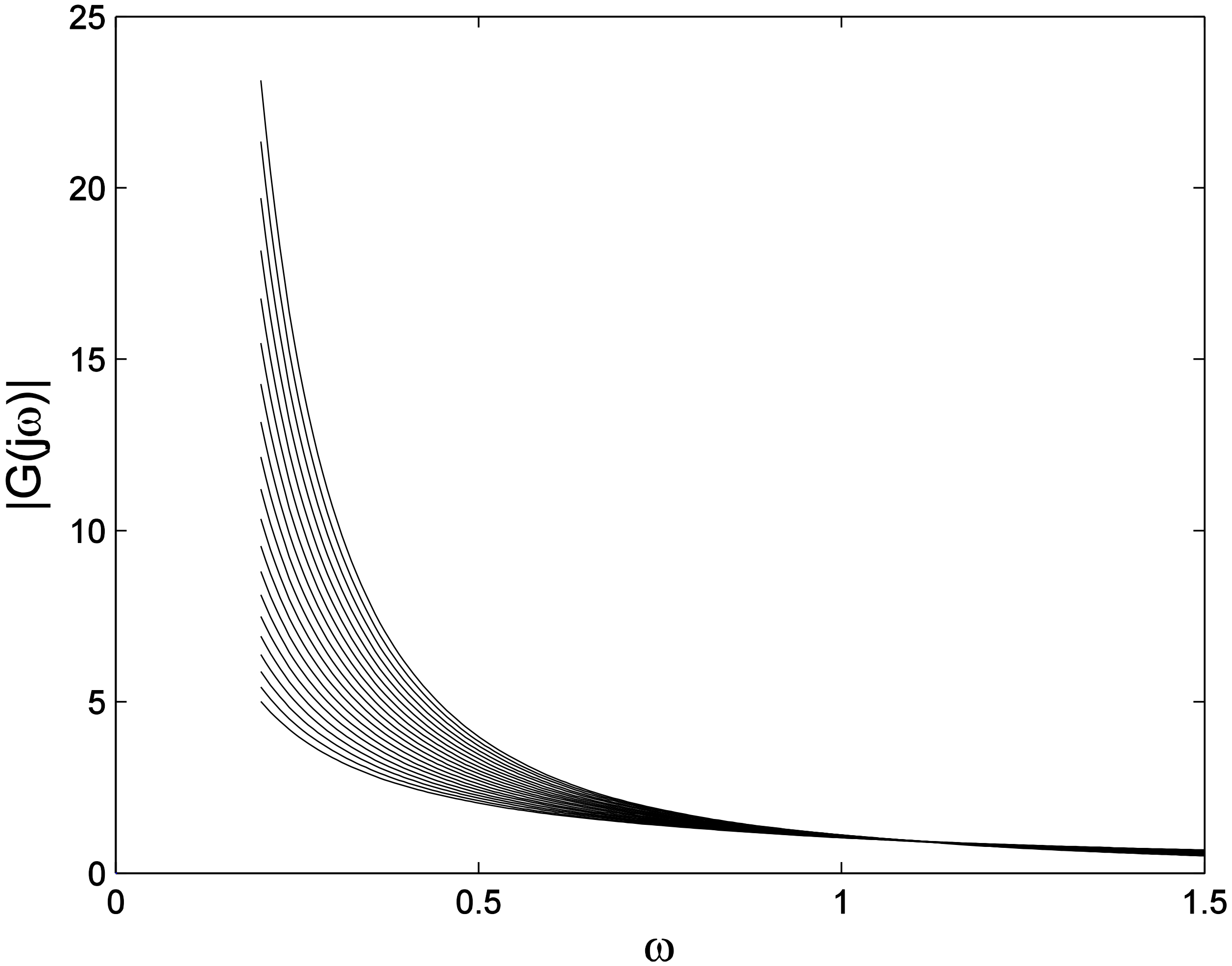

Fig.3.12 Amplitude characteristic of the FI for different orders $\nu \in [1.0, \cdots 1.95, 2.0]$ and constant $T_u = 1$

The phase angle characteristic of the FI

$$\varphi(\omega) = -\nu \operatorname{arctg}\left(\operatorname{tg}\frac{\omega}{2} \right)$$

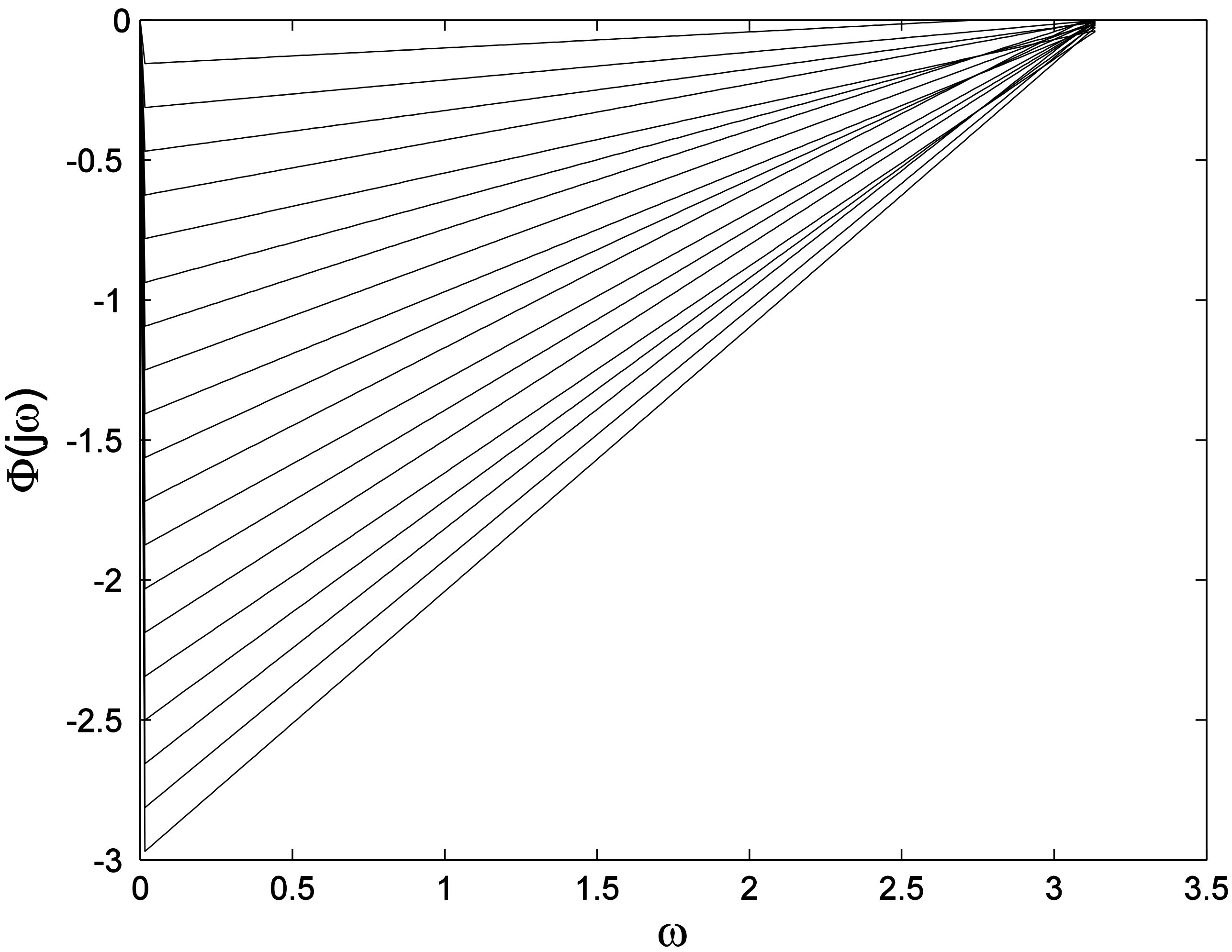

Fig.3.13 Phase angle characteristic of the FI for different orders $\nu \in [0, \cdots 1.95, 2.0]$ and constant $T_u = 1$

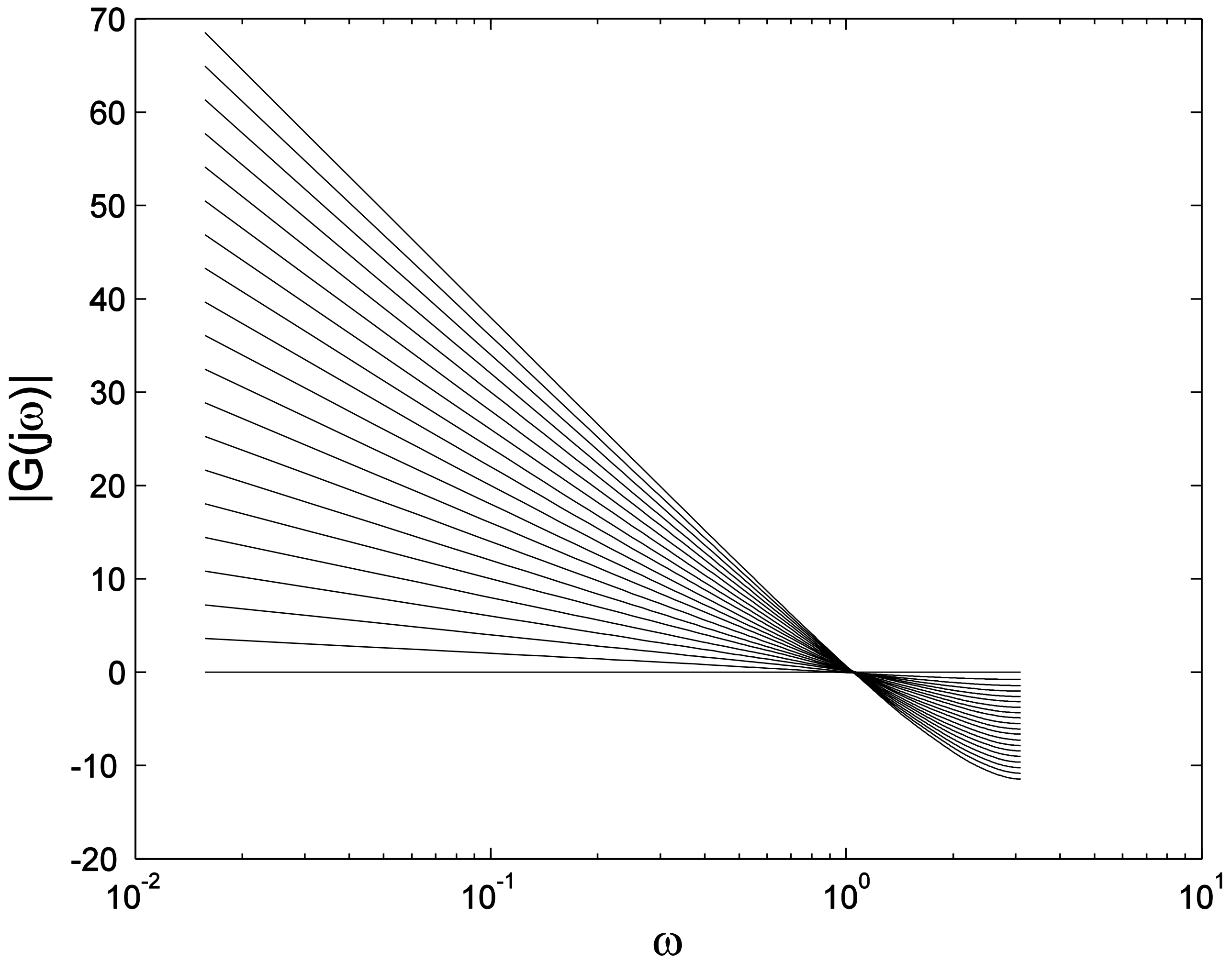

Fig.3.14 Bode amplitude characteristic of the FI for different orders $\nu \in [0.05, \cdots 1.95, 2.0]$ and constant $T_u = 1$

3.2 Frequency characteristics of the FO discrete inertial element (FOIE)

$$G_1^{(\nu)}(z) = \frac{1}{\left[T_u(1-z^{-1})\right]^{\nu}+1} = \frac{1}{\left(\dfrac{1-z^{-1}}{\omega_u}\right)^{\nu}+1}$$

$$G_1^{(\nu)}(e^{j\omega}) =$$

$$\frac{1}{\left[T_u(1-e^{-j\omega})\right]^{\nu}+1} = \frac{1}{\left(\dfrac{1-e^{-j\omega}}{\omega_u}\right)^{\nu}+1}$$

Nyquist plots of the FOIE

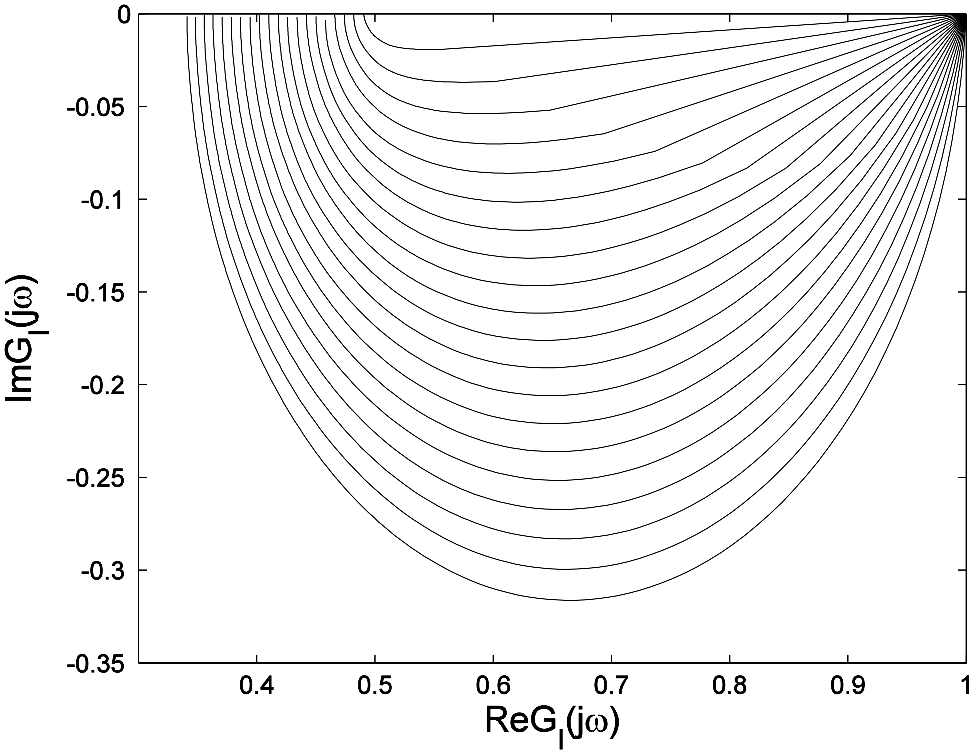

Fig.3.15 Nyquist plots of FOIE for $\nu \in [0.05, 0.1, \cdots 0.95, 1.0]$

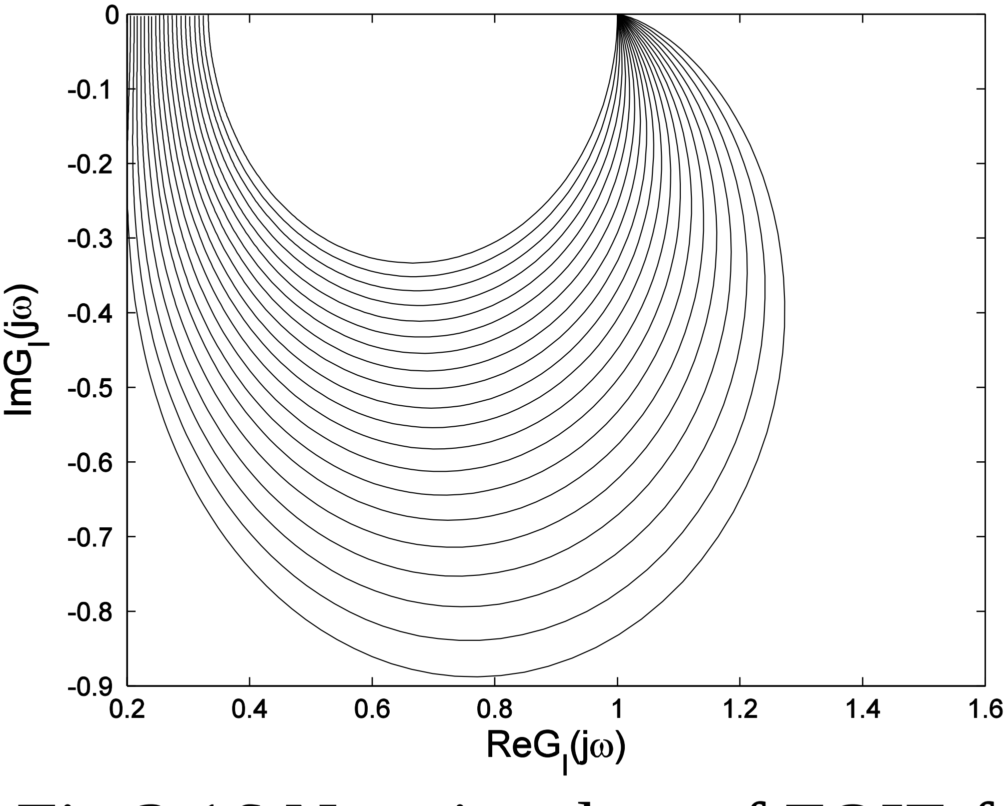

Fig.3.16 Nyquist plots of FOIE for $\nu \in [1.00, 1.08, \cdots 1.72, 1.80]$

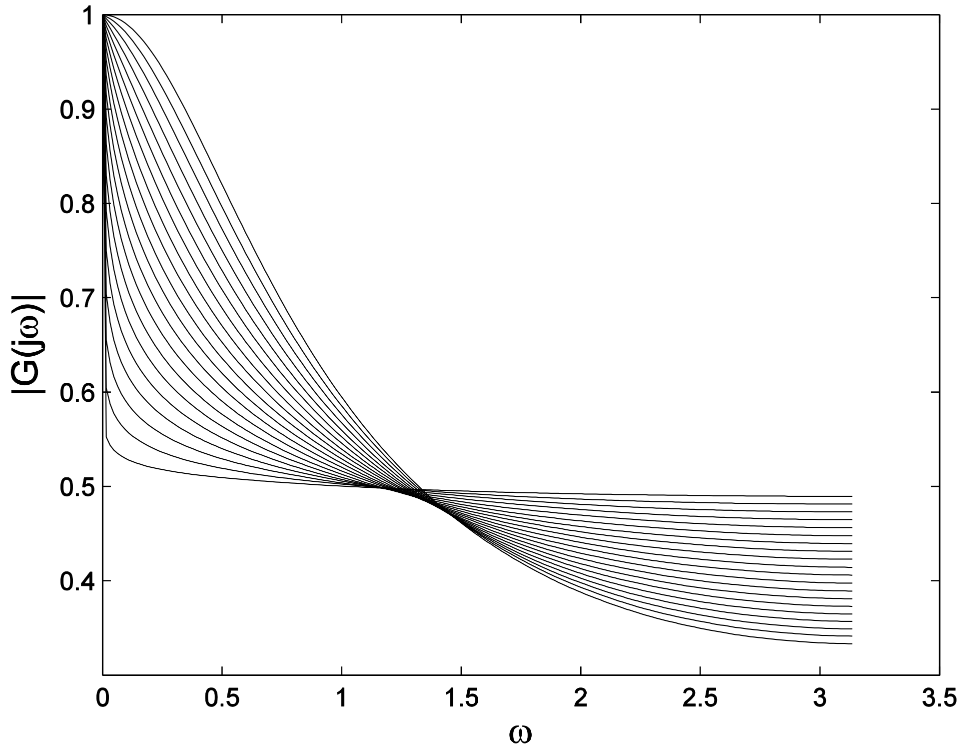

Fig.3.17 Amplitude characteristic of the FOIE for orders $\nu \in \left[0.1,\, 0.2,\, \cdots\, 0.9,\, 1.0\right]$ and constant T_u

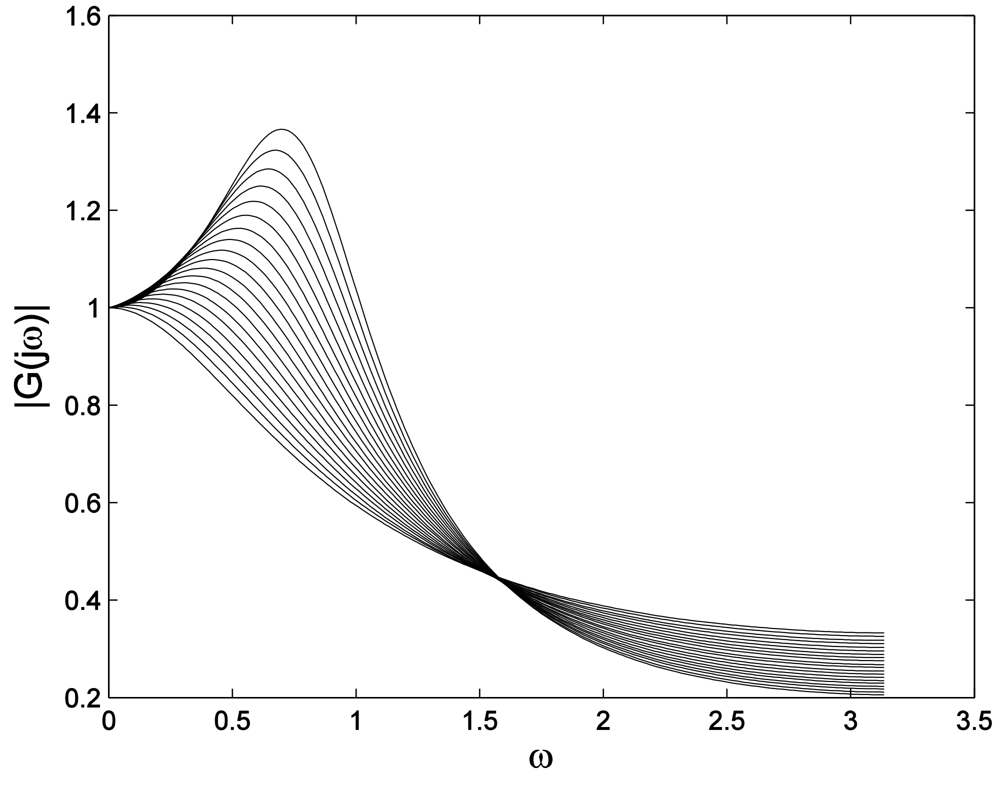

Fig.3.18 Amplitude characteristic of the FODIE for orders $\nu \in \left[1.1, 1.2, \cdots 1.8, 1.9\right]$ and constant T_u.

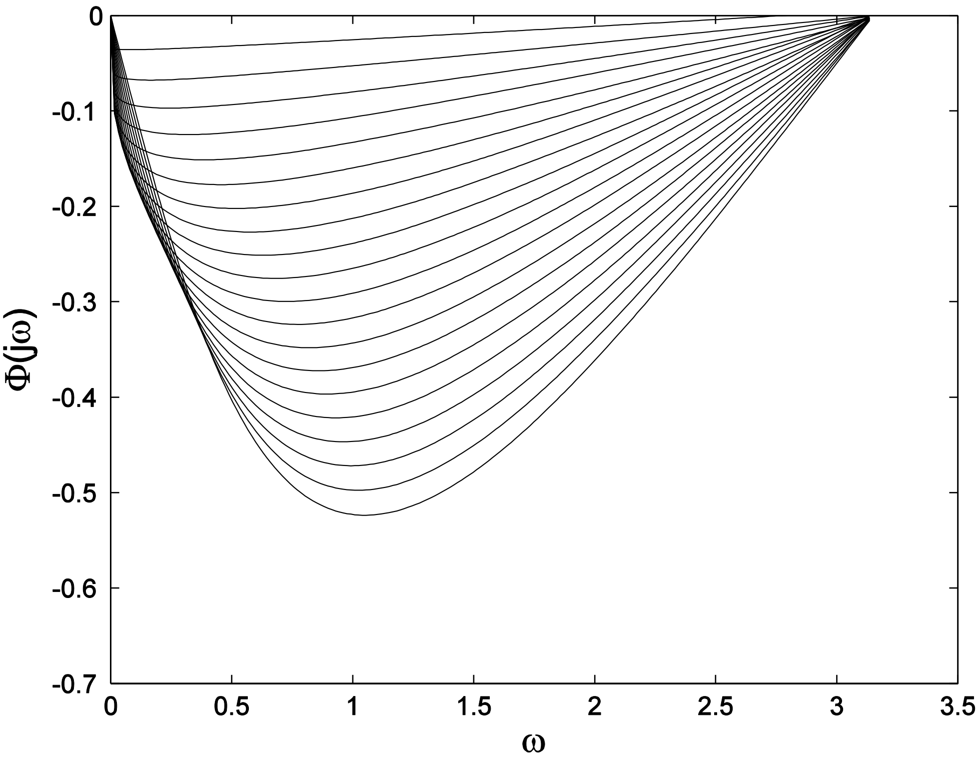

Fig.3.19 Phase angle characteristic of the FODIE for $\nu \in [0.1, 0.2, \cdots 0.9, 1.0]$ and constant T_u.

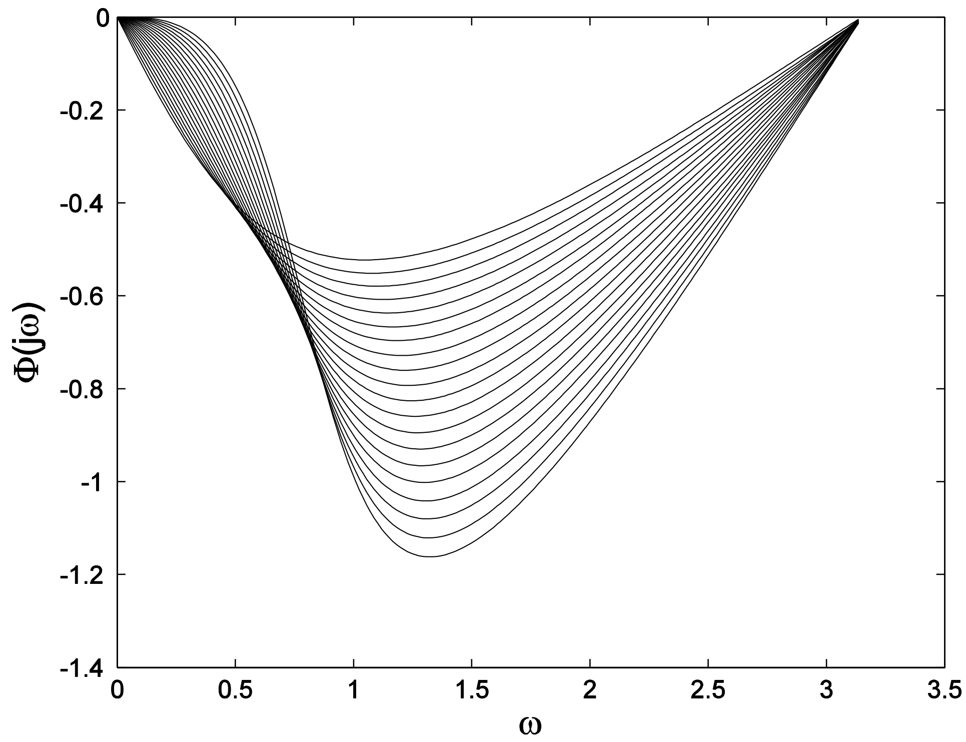

Fig.3.20 Phase angle characteristic of the ODIE for $\nu \in [1.1, 1.2, \cdots 1.8, 1.9]$ and constant T_u.